THE ALGEBRAIC STRUCTURE OF AN IMPLICIT RUNGE-KUTTA TYPE METHOD

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Abstract

In this paper, the theory of linear transformation (Homomorphism) and monomorphism is applied to a first order Runge-Kutta Type Method illustrated in a Butcher Table and the extended second order Runge- Kutta Type Methods in order to substantiate their uniform order and error constant.

Keywords:,linear transformation, implicit, runge-kutta type

1. Introduction

A function T between two vector spaces $T: V \to W$ that preserves the operations of addition if v_1 and $v_2 \epsilon V$ then

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad (1)$$

And scalar multiplication if $v\epsilon V$ and $r\epsilon R$, then

$$T(r.v) = rT(v) \quad (2)$$

is a homomorphism or linear Transformation, Agam (2013).

A homomorphism that is one to one or a mono is called a monomorphism.

The monomorphism Transformation preserves its algebraic structure and the order of the Domain into its Range.

Butcher and Hojjati (2005) extended the general linear method to the case in which second derivative as well as first derivative can be calculated. They constructed methods of third and fourth order which are A-stable, possess the Runge-Kutta stability property and have a diagonally implicit structure for efficient implementation. However, they concentrated on only linear problems for which it is possible to compute accurate starting methods, the general purpose starting methods for non-linear problems were not developed. Okunuga, Sofoluwe , Ehigie and Akanbi (2012) presented a direct integration of second order ordinary differential equation using only Explicit Runge-Kutta Nystrom (RKN) method with higher derivative. They derived and tested various numerical schemes on standard problems. Due to the limitations of Explicit Runge-Kutta (ERK) in handling stiff problems, the extension to higher order Explicit Runge-Kutta Nystrom

(RKN) was considered and results obtained showed an improvement over conventional Explicit Runge-Kutta schemes. The Implicit Runge-Kutta scheme was however not considered. Yahaya and Adegboye (2013) derived an implicit 6-stage block Runge-Kutta Type Method for direct integration of second order (special or general), third order (special or general) as well as first order initial value problem and boundary value problems. The theory of Nystrom was adopted in the reformulation of the methods. The convergence and stability analysis of the method were conducted and the region of absolute stability plotted. The method was A-stable, possessed the Runge-Kutta stability property, had an implicit structure for efficient implementation and produced simultaneously approximation of the solution of both linear and non linear initial value problem. Numerical results were obtained to illustrate the performance and comparisons made to some standard known. The step number was restricted to 4.

2. Methodology

Let T be a linear transformation which is continuously differentiable on a set of ordered three- tuple vector $\in \mathbb{R}^3$ as follows

$$V_i = (x + c_i h, y + \sum_{j=1}^{s} a_{ij}\, T(v_j), y' + \sum_{j=1}^{s} a_{ij} T'(v_j)) \in \mathbb{R}^3 \qquad (3)$$

$$T(V_i) = h(y' + \sum_{j=1}^{s} a_{ij} T'(v_j)) \qquad (4)$$

and

$$T'(v_i) = hf\big(x + c_i h, y + \sum_{j=1}^{s} a_{ij}\, T(v_j), y' + \sum_{j=1}^{s} a_{ij} T'(v_j)\big) = hm_i \qquad (5)$$

$$i{:}e\ m_i = f\big(x + c_i h, y + \sum_{j=1}^{s} a_{ij}\, T(v_j), y' + \sum_{j=1}^{s} a_{ij} T'(v_j)\big) \qquad (6)$$

Then the Transformation $T: \mathbb{R}^3 \to \mathbb{R}$ is a well defined monomorphism:

Proof

Let $u, v \in \mathbb{R}$ defined by

$$U = (x + c_i h, y_1 + \sum_{j=1}^{s} a_{ij}\, T(u_j), y_1' + \sum_{j=1}^{s} a_{ij} T'(u_j)) \qquad (7)$$

$$V = (x + c_i h, y_2 + \sum_{j=1}^{s} a_{ij}\, T(v_j), y_2' + \sum_{j=1}^{s} a_{ij} T'(v_j)) \qquad (8)$$

$$T(U + V) = h(y_1' + y_2' + \sum_{j=1}^{s} a_{ij}\, (T'(u_j) + T'(v_j)) \qquad (9)$$

By the definition of T on $\mathbb{R}^3$

$$= h\big(y_1' + \sum_{j=1}^{s} a_{ij} T'(u_j)\big) + h(y_2' + \sum_{j=1}^{s} a_{ij} T'(v_j)) \qquad (10)$$

$$T(U + V) = T(U) + T(V) \qquad (11)$$

$$T(k.U) = k.T(U) \qquad (12)$$

Hence T is a homomorphism

Now we show that T is $1-1$

Let $u, v \in \mathbb{R}^3$ with

$$T(u) = T(v) \tag{13}$$

By definition of T

$$\Rightarrow h(y_1' + \textstyle\sum_{j=1}^{s} a_{ij}\, T'(u_j)) = h(y_2' + \textstyle\sum_{j=1}^{s} a_{ij}\, T'(v_j)) \tag{14}$$

Since

$$T(u) = T(v) \text{ then } T(u_j) = T(v_j) \text{ and } T'(u_j) = T'(v_j) \tag{15}$$

$$y_1 = y_2 \text{ and } x_1 + c_i h = x_2 + c_i h \text{ i:e } x_1 = x_2 \tag{16}$$

Hence $U = V$ (17)

Thus T is $1-1 \Leftrightarrow$ a monomorphism from $\mathbb{R}^3 \to \mathbb{R}$

Remark: The necessity for the above proposition is to ensure that the algebraic structure and the order does not change during the transformation. We consider for the case K=1.

Consider the Butcher Table 3.1 and Table 3.2

Table 3.1: The Butcher Table for K=1

0	0	0	0
$\frac{1}{2}$	0	$\frac{3}{4}$	$\frac{-1}{4}$
1	0	1	0
	0	1	0

Table 3.2: The Butcher Table for second order for K=1

0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{3}{4}$	$\frac{-1}{4}$	0	$\frac{5}{16}$	$\frac{-3}{16}$
1	0	1	0	0	$\frac{3}{4}$	$\frac{-1}{4}$
	0	1	0	0	$\frac{3}{4}$	$\frac{-1}{4}$

The table 3.1 satisfies the Runge-Kutta conditions for solution of first order ode since

$$(i)\quad \sum_{j=1}^{s} a_{ij} = c_i \tag{18}$$

$$(ii)\quad \sum_{j=1}^{s} b_j = 1 \tag{19}$$

We consider the general second order differential equation in the form

$$y'' = f(x, y, y'),\ y(x_0) = y_0 \quad y'(x_0) = y_0' \tag{20}$$

$$y'' = f(v), \qquad v = (x, y, y') \tag{21}$$

$$T(V_i) = T\left(x + c_i h, y + \textstyle\sum_{j=1}^{3} a_{ij} T(V_j), y' + \sum_{j=1}^{3} a_{ij} T'(V_j)\right) \tag{22}$$

$$= h\,(y' + \textstyle\sum_{j=1}^{3} a_{ij}\, T'(V_j)) = h\,(y' + \sum_{j=1}^{3} a_{ij}\, hm_j) \tag{23}$$

$$T(V_1) = h(y' + 0hm_1 + 0hm_2 + 0hm_3) \tag{24}$$

$$T(V_2) = h(\left(y' + 0hm_1 + \tfrac{3}{4}hm_2 - \tfrac{1}{4}hm_3\right)) \tag{25}$$

$$T(V_3) = h(y' + 0hm_1 + hm_2 + 0hm_3) \tag{26}$$

Also

$$m_i = T'(V_j) = f\left(x + c_i h, y + \textstyle\sum_{j=1}^{s} a_{ij} T(V_j), y' + \sum_{j=1}^{s} a_{ij} T'(V_j)\right) \tag{27}$$

$$m_1 = f(x + 0h, y + 0 + 0 + 0) \tag{28}$$

$$m_2 = f(x + \tfrac{1}{2}h, y + \tfrac{1}{2}hy' + \tfrac{5}{16}h^2 m_2 - \tfrac{3}{16}h^2 m_3, y' + \tfrac{3}{4}hm_2 - \tfrac{1}{4}hm_3) \tag{29}$$

$$m_3 = f(x + h, y + hy' + \tfrac{3}{4}h^2 m_2 - \tfrac{1}{4}h^2 m_3, y' + hm_2\,) \tag{30}$$

The direct method for solving $y'' = f(x, y, y')$ is now

$$y_{n+1} = y_n + b_1 T(V_1) + b_2 T(V_2) + b_3 T(V_3) \tag{31}$$

$$y_{n+1} = y_n + 0T(V_1) + T(V_2) + 0T(V_3) \tag{32}$$

$$y_{n+1} = y_n + T(V_2) \tag{33}$$

$$y_{n+1} = y_n + h\left(y_n' + \tfrac{3}{4}hm_2 - \tfrac{1}{4}hm_3\right) \tag{34}$$

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{4}(3m_2 - m_3) \tag{35}$$

$$y_{n+1}' = y_n' + b_1 T'(V_1) + b_2 T'(V_2) + b_3 T'(V_3) \tag{36}$$

$$y_{n+1}' = y_n' + 0hm_1 + hm_2 + 0hm_3 \tag{37}$$

$$y_{n+1}' = y_n' + hm_2 \tag{38}$$

3. Results and Discussion

We made use of the coefficients of the Butcher table of the first order RKTM to prove to the second order RKTM. Equation (35) and (38) satisfy the Runge-Kutta consistency conditions of second and first order respectively. This further shows that it is a monomorphism.

4. Conclusion

This research work established the reason behind uniform order and error constant of the first order Runge-kutta type method and the extended second order Runge-kutta type method. Also why the algebraic structure and the order of the two methods are preserved and not changed during the transformation.

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