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COA-SOWUNMI'S LEMMA AND ITS APPLICATION TO THE STABILITY ANALYSIS OF EQUILIBRIUM STATES OF THE NON-LINEAR AGE-STRUCTURED POPULATION MODEL

[1]Akinwande, N.I., [2]Somma, S. A.

Department of Mathematics & Statistics, Federal University of Technology, P.M.B 65, Minna, Nigeria

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Abstract: **In this work, we present a result in the form of a lemma which we name COA-Sowunmi's Lemma, its proof and application to the stability analysis of the transcendental characteristics equation arising from the perturbation of the steady state of the non-linear age-structured population model of Gurtin and MacCamy [11]. Necessary condition for the asymptotic stability of the equilibrium state of the model is obtained in the form of constrained inequality on the vital parameters of the model. The result obtained is then compared with that of an earlier work by the [4].**

Keywords: **Characteristic equation, Steady or Equilibrium state, Stability.**

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1. INTRODUCTION

The analysis of the stability or otherwise of the equilibrium or steady states of mathematical models of population dynamics forms the core of our research works over the years with Prof. C.O.A. Sowunmi for several years. [1] – [6], [12] and [13]

The major problem is usually to obtain the roots of the characteristic equations which usually arise from the perturbation of the equilibrium states of the dynamical system explicitly but often such would come up implicitly. This is a routine mathematical problem when the equation is a polynomial. However the equations obtained often are of the type classified as transcendental. Professor Sowunmi had some intuitive ideas of the methods of analyzing such roots. He shared these ideas with me many times as my supervisor for my Masters and Doctoral programmes, and in 2005 when he was invited to the Department of Mathematics, Federal University of Technology, Minna as an external examiner.

Two of such ideas proposed and discussed with him have been successfully transformed into Mathematical theorem/lemma by this author in [1], [2], [5], [6] and are currently being applied by Mathematical Modellers of dynamical systems to analyze transcendental characteristic equations for stability. It is one of them that I am presenting and dedicating to his name in this work which has been presented in some international confeferences..

We present the result in the form of a Lemma with the proof. The Lemma is applied to the stability analysis of the transcendental characteristic equation arising from the perturbation of the steady states of the Non-Linear Age-Structured Population Model of Gurtin and Mac Camy [11]. The application of the lemma gives necessary conditions for the local asymptotic stability of the equilibrium state.

2. LEMMA (COA-SOWUNMI'S LEMMA)

Suppose the characteristic equation arising from the perturbation of the equilibrium state of a dynamical system is of the transcendental form

$$\int_0^\infty e^{-\lambda t}F(t)dt=1 \tag{1}$$

where λ is the eigenvalue and $F(t)$ is some continuous function of t. Let

$$g(u)=\int_0^\infty e^{-ut}|F(t)|dt \tag{2}$$

be the real part of the expression obtained from the left hand side of (1) by setting $\lambda = u \pm iv$, then a necessary condition for the local asymptotic stability of the equilibrium state of the system is given by

$$g(0)<1 \tag{3}$$

The equilibrium or steady state is unstable if

$$g(0)>1 \tag{4}$$

Proof of the Lemma:

Let $\lambda = u \pm iv$, then equation (1) gives the pair of equations:

$$\int_0^\infty e^{-ut}F(t)\cos vt dt=1 \tag{5}$$

and

$$\int_0^\infty e^{-ut}F(t)\sin vt dt=0 \tag{6}$$

From stability theories, [1], [2], [6] – [10], a necessary and sufficient condition for local asymptotic stability is for the real part of the eigenvalue to be in the negative half plane, i.e. for Re $\lambda = u < 0$.

From the real part equation (5) we derive the function $g(u)$ given by

$$g(u)=\int_0^\infty e^{-ut}|F(t)|dt \tag{7}$$

Which is an exponentially monotone decreasing function of u. Furthermore, every solution of the equation (7) is less or equal to the solution of the equation (5). The solution u_c will be negative if

$$\int_0^\infty |F(t)|dt<1 \tag{8}$$

and positive if

$$\int_0^\infty |F(t)|dt>1 \tag{9}$$

Note that if $F(t)>0$, then $F(t)=|F(t)|$.

This is illustrated graphically below.

Suppose the graphs of the function $g(u)$ and the constant function $y = 1$ intersect at the point $(u_c, 1)$, if $g(0) < 1$ then $u_c < 0$; as shown in Figure 1 while if $g(0) > 1$ it follows that $u_c > 0$; as shown in Figure 2.

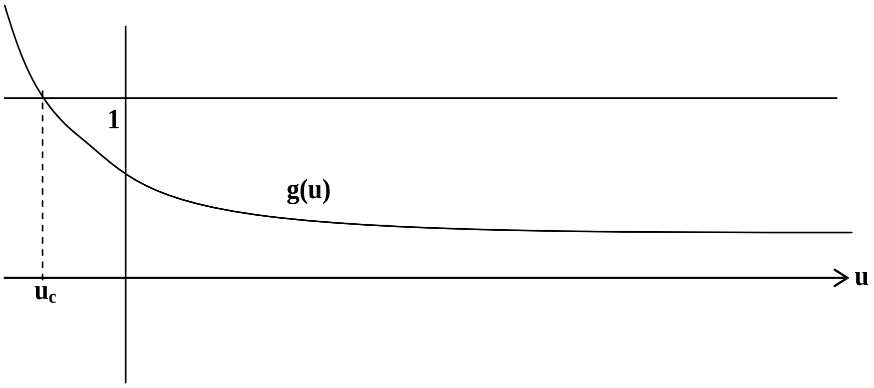

Figure 1: $g(0) < 1 \Rightarrow u_c < 0$

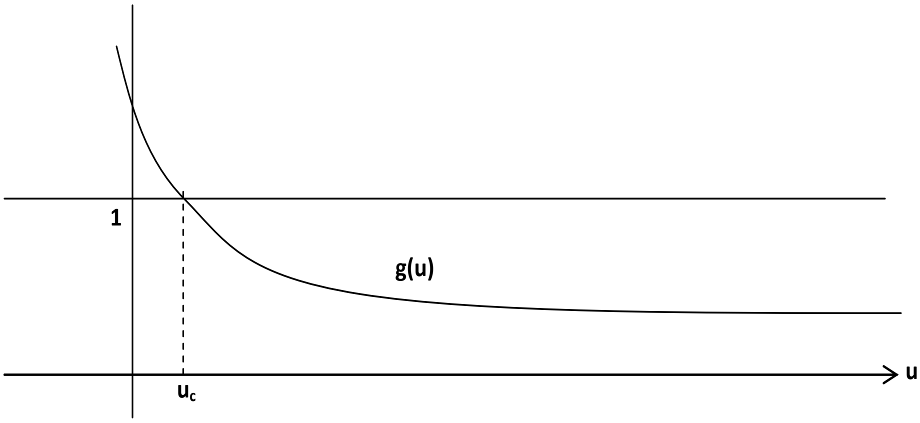

Figure 2: $g(0) > 1 \Rightarrow u_c > 0$

3. THE MODEL EQUATION AND THE EQUILIBRIUM STATE

We define:

$\beta(.,\ .)$ = birth modulus.

$\mu(.,\ .)$ = death modulus.

$\lambda = x + iy$ = the eigenvalue; i.e. root of the characteristics equation.

Ω = the maximum life-span of the population. (in [9], $\Omega = a^* < \infty$)

$\rho(t, a)$ = the population of those aged a at time t.

$\phi(a)$ = the equilibrium state population of those aged a.

The equations of the non-linear population model of Gurtin and MacCamy [11] are given by equations (10)-(13) below:

$$\frac{\partial\rho(t,\ a)}{\partial t}+\frac{\partial\rho(t,\ a)}{\partial a}+\mu(P(t),\ a)\rho(t,\ a)=0 \qquad (10)$$

$$\rho(t,\ 0)=B(t)=\int_0^{\Omega}\beta(P(t),\ a)\rho(t,\ \mathrm{a})da \qquad (11)$$

$$\rho(0,\ \mathrm{a})=\phi(a) \qquad (12)$$

$$P(t)=\int_0^{\Omega}\rho(t,\ a)da \qquad (13)$$

At equilibrium state let:

$$\rho(t,\ a)=\phi(a);\ P(t)=P_0=\int_0^{\Omega}\phi(a)da \qquad (14)$$

$$\mu(P(t),\ a)=\mu(P_0,\ a)=\mu_0(a);\ \ \beta(P(t),\ a)=\beta(P_0,\ a)=\beta_0(a) \qquad (15)$$

$$\mu p(P_0,\ a)=\mu_0'(a);\ \ \beta p(P_0,\ a)=\beta_0'(a) \qquad (16)$$

$$\phi(0)=B(0)=\int_0^{\Omega}\beta_0(a)\phi(a)da=B_0 \qquad (17)$$

Thus, $\phi(.)$ satisfies the equation:

$$\frac{d\phi}{da}+\mu_0(a)\phi(a)=0 \qquad (18)$$

And the solution of (9) is given by

$$\phi(a)=\phi(0)\exp\left\{-\int_0^{a}\mu_0(a)d\lambda\right\} \qquad (19)$$

From (17) and (19),

$$\phi(0)=\int_0^{\Omega}\beta_0(a)\phi(0)\exp\left\{-\int_0^{a}\mu_0(\lambda)d\lambda\right\}da$$

i.e.

$$\int_0^{\Omega}\beta_0(a)\exp\left\{-\int_0^{a}\mu_0(\lambda)d\lambda\right\}da=1 \qquad (20)$$

If we define:

$$\pi(a-s)=\exp\left\{-\int_0^{a}\mu_0(\lambda)d\lambda\right\} \qquad (21)$$

Then (19) and (20) respectively become:

$$\phi(a)=B_0\pi(a) \qquad (22)$$

$$\int_0^{\Omega}\beta_0(a)\pi(a)da=1 \qquad (23)$$

and

$$P_0=\beta_0\int_0^{\Omega}\pi(a)da \qquad (24)$$

3.1. Perturbation of Equilibrium State and the Characteristics Equation:

Let

$$\rho(t,\ a)=\phi(a)+\eta(a)e^{\lambda t};\ P(t)=P_0+P(t) \tag{25}$$

$$p(t)=e^{\lambda t}\int_0^{\Omega}\eta(a)da=\bar{p}e^{\lambda t};\quad \bar{p}=\int_0^{\Omega}\eta(a)da \tag{26}$$

Neglecting second and higher order terms of $\eta(a)$ and $p(t)$; $\eta(a)$ satisfies the following:

$$\frac{d\eta(a)}{da}+[\lambda+\mu_0(a)]\eta(a)+B_0\bar{p}f(a)=0 \tag{27}$$

where

$$f(a)=\mu_0'(a)\pi(a) \tag{28}$$

and

$$\eta(0)=\eta_0=\int_0^{\Omega}\beta_0{}'(a)\eta(a)da+kB_0\bar{p} \tag{29}$$

where

$$k=\int_0^{\Omega}\beta_0'(a)\pi(a)da \tag{30}$$

The solution of (27) is then given by:

$$\eta(a)=\eta_0\exp\left\{-\int_0^a[\lambda+\mu_0(\lambda)]d\lambda\right\}$$

$$-B_0\bar{p}\int_0^a f(a)\exp\left\{-\int_s^a[\lambda+\mu_0(\lambda)]d\lambda\right\}ds \tag{31}$$

i.e.

$$\eta(a)=\eta_0\pi(a)\left[e^{-\lambda a}-1\right]$$

$$-B_0\bar{p}\int_0^a f(s)\pi(a-s)\left[e^{-\lambda a}-e^{-\lambda s}\right]ds \tag{32}$$

Now, using (32) in (29), we have:

$$\eta_0=kB_o\bar{p}+\eta_0\int_0^{\Omega}\beta_0(a)\pi(a)\left[e^{-\lambda a}-1\right]da$$

$$-B_0\bar{p}\int_0^{\Omega}\int_0^a B_0(a)f(s)\pi(a-s)\left[e^{-\lambda a}-e^{-\lambda s}\right]dsda \tag{33}$$

And using (32) in (26), $\bar{p}$ will be given by:

$$\bar{p}=\eta_0\int_0^{\Omega}\pi(a)\left[e^{-\lambda a}-1\right]da$$

$$-B_0\bar{p}\int_0^{\Omega}\int_0^a f(s)\pi(a-s)\left[e^{-\lambda a}-e^{-\lambda s}\right]dsda \tag{34}$$

Let

$$A = \int_0^{\Omega} \beta_0 \pi(a)\left[e^{-\lambda a} - 1\right] da \quad (35)$$

$$B = B_0 \int_0^{\Omega} \int_0^{a} B_0(a) f(s) \pi(a-s)\left[e^{-\lambda a} - e^{-\lambda s}\right] ds da \quad (36)$$

$$C = \int_0^{\Omega} \pi(a)\left[e^{-\lambda a} - 1\right] da \quad (37)$$

$$D = B_0 \int_0^{\Omega} \int_0^{a} f(s) \pi(a-s)\left[e^{-\lambda a} - e^{-\lambda s}\right] ds da \quad (38)$$

Then (33) and (34) will respectively take the form:

$$\eta_0 = kB_o \bar{p} + A\eta_0 + B\bar{p} \quad (39)$$

$$\bar{p} = C\eta_0 + D\bar{p} \quad (40)$$

Eliminating η_0 and $\bar{p}$ from (39) and (40) gives:

$$(1-A)(1+D) + (B - kB_o)C = 0 \quad (41)$$

which gives the characteristics (transcendental) equation. Substituting back for A, B, C and D as given by equations (35) and (38) respectively gives:

$$\left\{1 - \int_0^{\Omega} \beta_0 \pi(a)\left[e^{-\lambda a} - 1\right] da\right\}\left\{B_0 \int_0^{\Omega} \int_0^{a} f(s) \pi(a-s)\left[e^{-\lambda a} - e^{-\lambda s}\right] ds da\right\} +$$

$$B_0 \left\{-k + \int_0^{\Omega} \int_0^{a} B_0(a) f(s) \pi(a-s)\left[e^{-\lambda a} - e^{-\lambda s}\right] ds da\right\} \int_0^{\Omega} \pi(a)\left[e^{-\lambda a} - 1\right] da = 0 \quad (42)$$

We note that

$$f(s)\pi(a-s) = \mu_0'(s)\pi(s)\pi(a-s) = \mu_0'(s)\pi(a) \quad (43)$$

Let

$$\bar{\pi} = \int_0^{\Omega} \pi(a) da \quad (44)$$

Using equations (23), (43) and (44), equation (42) becomes:

$$\left\{2 - \int_0^{\Omega} \beta_0(a)\pi(a) e^{-\lambda a} da\right\}\left\{1 + B_0 \int_0^{\Omega} \int_0^{a} \mu_0'(s) \pi(a)\left[e^{-\lambda a} - e^{-\lambda s}\right] ds da\right\} +$$

$$B_0 \left\{-k + \int_0^{\Omega} \int_0^{a} \beta_0(a) \mu_0'(s) \pi(a)\left[e^{-\lambda a} - e^{-\lambda s}\right] ds da\right\}\left\{\int_0^{\Omega} \pi(a) e^{-\lambda a} da - \bar{\pi}\right\} = 0 \quad (45)$$

Equation (45) is the required characteristics equation.

3.2 Stability Condition for the Steady State:

Comparing the characteristic equation with the equations in the lemma using equations (6) and (41),

$$g(u) = \int_0^{\Omega} \beta_0(a) e^{-ua} \pi(a) da + B_0 \left[\int_0^{\Omega} \beta_0(a) e^{-ua} \pi(a) da\right]\left[\int_0^{\Omega}\int_0^{a} e^{-u(a-s)} \pi(a-s) f(s) ds da\right]$$

$$+ B_0 \int_0^{\Omega} e^{-ua} \pi(a) da \left[k + \int_0^{\Omega} \beta_0(a) \int_0^{a} e^{-u(a-s)} \pi(a-s) f(s) ds da\right] - B_0 \int_0^{\Omega}\int_0^{a} e^{-u(a-s)} \pi(a-s) f(s) ds da \quad (46)$$

and

$$g(0)=\int_0^{\Omega}\beta_0(a)\pi(a)da+B_0\left[\int_0^{\Omega}\beta_0(a)\pi(a)da\right]\left[\int_0^{\Omega}\int_0^{a}\pi(a-s)f(s)dsda\right]$$

$$+B_0\int_0^{\Omega}\pi(a)da\left[k+\int_0^{\Omega}\beta_0(a)\int_0^{a}\pi(a-s)f(s)dsda\right]-B_0\int_0^{\Omega}\int_0^{a}\pi(a-s)f(s)dsda \qquad (47)$$

using the identities (26) and (27), the inequality (43) takes the form:

$$g(0)=1+P_0\left[k+\int_0^{\Omega}\beta_0(a)\int_0^{a}\pi(a-s)f(s)dsda\right] \qquad (48)$$

hence the required sufficient condition for the asymptotic stability of the equilibrium state is the inequality constraint

$$k+\int_0^{\Omega}\beta_0(a)\int_0^{a}\pi(a-s)f(s)dsda<0 \qquad (49)$$

substituting for k and f into (45) as given from (30) and (32) gives

$$\int_0^{\Omega}\beta'_0(a)\pi(a)da+\int_0^{\Omega}\beta_0(a)\pi(a)\int_0^{a}\mu'_0(s)dsda<0 \qquad (49)$$

4. CONCLUSION

The result proposed in this work enables us to obtain sufficient condition for the asymptotic stability of the steady state. In [11] the method used led to the condition

$$\beta'_0\le\mu'_0 \qquad (50)$$

while in [4] the author obtained the constraint

$$\beta'_0\le\mu'_0\le 0 \qquad (51)$$

The current study has given integral constraint in terms of these two salient parameters. Clearly the integral inequality (49) is consistent with earlier results. Furthermore, it is more robust as it involves the birth and death modulli with their first derivatives with respect to the total population.

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