Development and Implementation of Block Unification Multi-step Methods for the Solution of Second Order Ordinary Differential Equations

View the article online for updates and enhancements.

IOP ebooks™

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the collection - download the first chapter of every title for free.

Development and Implementation of Block Unification Multi-step Methods for the Solution of Second Order Ordinary Differential Equations

Umaru Mohammed[1], Oyelami Oyewole[1], Mikhail Semenov[2] and Aliyu Ma'ali[3]

[1] Department of Mathematics, Federal University of Technology, Minna, Nigeria
[2] School of Nuclear Science & Engineering, Tomsk Polytechnic University, Russia
[3] Department of Mathematics and Computer Science, Ibrahihim Badamasi Babangida University, Lapai, Niger State, Nigeria

E-mail: [1]`umaru.mohd@futminna.edu.ng`, [2]`sme@tpu.ru`

Abstract. In this paper, linear multi-step hybrid block methods with three-, four- and five-step numbers are developed for approximating directly the solution of second order Initial and Boundary Value Problems (IBVPs). Multiple finite difference formulas are derived and combined in a block formulation to form a numerical integrator that provides direct solution to second order IBVPs over sub-intervals. A new class of orthogonal polynomials constructed as basis function to develop the hybrid block methods adopting collocation technique with a non-negative weight function. The scheme is applied as simultaneous integrator to second order initial value and boundary value problems of ODEs. The properties and convergence of the proposed method are discussed. The derived schemes were used to solve some problems and the numerical result shows the effectiveness, accuracy and superiority of the method over the existing methods found in the literature.

1. Introduction

It is the purpose of this work to construct a multistep hybrid linear method (HLM) for the numerical solution of initial value problems (IVPs) and boundary value problems (BVPs) in second order ordinary differential equations (ODEs):

$$y'' = f(x, y, y'), \quad x \in [a, b] \tag{1}$$

with initial conditions

$$y(a) = \alpha, \quad y(b) = \beta \quad \text{or} \tag{2a}$$

$$y'(a) = \alpha, \quad y'(b) = \beta \quad \text{or} \tag{2b}$$

$$y(a) = \alpha, \quad y'(b) = \beta. \tag{2c}$$

Even if Equations (1)-(2) can be transformed into a first-order system of double dimension, the development of numerical methods for its direct integration seems more natural and efficient [1]. The direct method which are self-starting and take less computation time are develop in terms of

linear multistep methods (LMMs) [2] which are called block method. The collocation technique to define the parameters of the methods was introduced by Hairer and Wanner [3] for the first-order equation. In the paper [1] this technique extending in second-order case. In the paper [4] authors used the self-starting scheme to derive a class of one-step hybrid methods for the numerical solution of second order differential equation with power series. In the study [5] a family of second derivative block methods for stiff initial value problems (IVPs) in ODEs is proposed. In the paper [6] authors presented a Block Nyström method that is applied to directly solve the Lane-Emden type equations which belong to a class of nonlinear singular ODEs.

In this paper, we applied derivation of numerical method based on the idea of collocation and interpolation procedures using the nonnegative weight function to solve second order ODEs.

Our derived scheme yield very good results compared to the existing methods in the literature [7, 8, 9, 10, 11, 12] and it is also able to solve IVPs and BVPs.

The paper is organized as follows. In Section 2, we derive a continuous approximation for the exact solution. The stability and convergence properties of proposed methods are analyzed in Section 4. A brief discussion of numerical results presented in Section 5.

2. Derivation of the Methods
2.1. Construction of Orthogonal Polynomial Basis Function
Two functions are said to be orthogonal to one another if their inner product is zero [13, 14], hence a family of functions forms an orthogonal system on an interval (a, b) with a weight function $w(x)$ if for any two distinct members of the family

$$\langle \varphi_1, \varphi_2 \rangle_w = \int_a^b \varphi_1(x)\varphi_2(x)w(x)dx = 0. \tag{3}$$

An orthogonal system can be written as a sequence of functions $\{\varphi_n\}_{n=0}^{\infty}$ and the corresponding orthogonal property can be expressed as

$$\langle \varphi_i, \varphi_j \rangle_w \quad \begin{cases} > 0, & i = j \\ = 0, & i \neq j. \end{cases}$$

We defined the orthogonal polynomials $\varphi_i(x)$ over the interval $(-1, 1)$ with respect to the nonnegative weight function $w(x) = x^2$ as

$$\varphi_i(x) = \sum_{i=0}^{n} C_i^n x^i. \tag{4}$$

In order to calculate the real coefficients $C_i^n \in \mathbb{R}$ we use the additional (normalization) property

$$\varphi_i(1) = 1, \quad i = 0, 1, \dots, n. \tag{5}$$

Following the study [12], recall that A_n denotes the set of indexes for two distinct members of the orthogonal system:

$$A_n = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i + 1 \leq j \leq n\}. \tag{6}$$

The set A_n is the finite set and depends on the value of n. Fixing $n = 8$ in (4) the following orthogonal system is obtained:

$$\varphi_0(x) = 1, \quad \varphi_1(x) = x, \quad \varphi_2(x) = \frac{1}{2}\left(5x^2 - 3\right), \quad \varphi_3(x) = \frac{1}{2}\left(7x^3 - 5x\right), \tag{7a}$$

$$\varphi_4(x) = \frac{1}{8}\left(63x^4 - 70x^2 + 15\right), \qquad \varphi_5(x) = \frac{1}{8}\left(99x^5 - 126x^3 + 35x\right),$$

$$\varphi_6(x) = \frac{1}{16}\left(429x^6 - 693x^4 + 315x^2 - 35\right), \varphi_7(x) = \frac{1}{16}\left(715x^7 - 1287x^5 + 693x^3 - 105x\right),$$

$$\varphi_8(x) = \frac{1}{128}\left(12155x^8 - 25740x^6 + 18018x^4 - 4620x^2 + 315\right).$$

These polynomials are employed as the basis functions for the derived scheme. In the same vein, the orthogonal polynomials $\varphi_i(x)$ $\forall i$ can be obtained [13].

We seek to derive numerical scheme using the LMMs form [2]:

$$\sum_{i=k-2}^{k} \alpha_i y_{n+1} = h^2 \sum_{i=0}^{k} \beta_i f_{n+1} + h^2 \beta_v f_v, \tag{8}$$

where α_i, β_i and β_v are unknown real constant and v must not be as an integers, h is a step of method. It is important to know that $\alpha_k = 1$, $\beta_k \neq 0$, α_0 and β_0 are non zero, $k = 3, 4, 5$ is the step's number of the proposed method.

We express the approximation of the analytic solution with a polynomial of the form

$$y(x) = \sum_{i=0}^{r+s-1} \alpha_i \phi_i(x), \tag{9}$$

where $\{\phi_i(x)\}_{i=0}^{r+s}$ is the orthogonal system, r is the number of collocation points, s is the number of interpolation points. The continuous approximation is then constructed with the imposition of two conditions in next equations:

$$y(x_{n+j}) = y_{n+j}, \quad j = 0, 1, 2, \ldots, r-1, \quad y''(x_{n+j}) = f_{n+j}, \quad j = 0, 1, 2, \ldots, s-1. \tag{10}$$

Equation (10) result to a $(r+s)$-system of equations which can be evaluated for solution through the matrix inversion algorithm. This is with a view to obtaining values for α_i. The explicit view of α_i, $i = 0, 1, \ldots, 5$ were obtained in the study [12] by the matrix inversion algorithm. The construction of final approximation is executed through the substitution of the values of α_i into equation (9). The method of continuous approximation can be adequately expressed as

$$y(x) = \sum_{j=k-2}^{k} \alpha_i(x) y_{n+i} + h^2 \sum_{i=0}^{k} \beta_i(x) y''_{n+i} + h^2 \beta_v(x) f_v, \tag{11}$$

where $\alpha_i(x)$, $\beta_i(x)$, and $\beta_v(x)$ are continuous coefficients. Denote the first derivatives (11) is as follows

$$y'(x) = \frac{1}{h} \left(\sum_{i=k-2}^{k} \alpha_i'(x) y_{n+i} + h^2 \sum_{i=0}^{k} \beta_i'(x) y''_{n+i} + h^2 \beta_v'(x) f_v \right), \tag{12}$$

and later we will use the notation $y'_{n+j} = z_{n+j}$ also.

2.2. Three-step Hybrid Linear Method with One Off-step Point (3HLM)

We use Equation (11) to obtain a 3HLM with the following specification: $r = 3$, $s = 4$, $k = 3$, $v = 5/2$, $\beta_v(x)$ and $\alpha_j(x)$, $\beta_j(x)$ for $j = 0, 1, \ldots, k$, can be expressed as functions of t, given that $t = \frac{x - x_n}{h}$ to obtain the continuous form as follows

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+5/2}. \tag{13}$$

Initial value problem. Evaluate equation (13), and evaluating at three points $\{n, n+\frac{5}{2}, n+3\}$ yields the following implicit discrete scheme:

$$y_n = 2y_{n+1} - y_{n+2} + \frac{23}{300}h^2 f_n + \frac{13}{15}h^2 f_{n+1} - \frac{1}{60}h^2 f_{n+2} + \frac{8}{75}h^2 f_{n+\frac{5}{2}} - \frac{1}{30}h^2 f_{n+3}, \tag{14a}$$

$$y_{n+\frac{5}{2}} = -\frac{1}{2}y_{n+1} + \frac{3}{2}y_{n+2} - \frac{21}{6400}h^2 f_n + \frac{223}{3840}h^2 f_{n+1} + \frac{457}{1280}h^2 f_{n+2} - \frac{31}{600}h^2 f_{n+\frac{5}{2}} + \frac{19}{1280}h^2 f_{n+3}, \tag{14b}$$

$$y_{n+3} = -y_{n+1} + 2y_{n+2} - \frac{1}{150}h^2 f_n + \frac{7}{60}h^2 f_{n+1} + \frac{11}{15}h^2 f_{n+2} + \frac{8}{75}h^2 f + \frac{1}{20}h^2 f_{n+3}. \tag{14c}$$

Differentiating the continuous scheme with respect to x and evaluating at x_n, x_{n+1}, x_{n+2}, $x_{n+5/2}$, x_{n+3} yields the following discrete scheme:

$$hz_n = y_{n+1} - y_{n+2} - h^2 \left(\frac{581}{1800}f_n - \frac{13}{10}f_{n+1} - \frac{19}{40}f_{n+2} - \frac{12}{250}f_{n+\frac{5}{2}} + \frac{23}{180}f_{n+3} \right), \tag{15a}$$

$$hz_{n+1} = -y_{n+1} + y_{n+2} + h^2 \left(\frac{1}{100}f_n - \frac{107}{360}f_{n+1} - \frac{11}{30}f_{n+2} - \frac{1}{24}f_{n+\frac{5}{2}} + \frac{44}{225}f_{n+3} \right), \tag{15b}$$

$$hz_{n+2} = -y_{n+1} + y_{n+2} + h^2 \left(\frac{13}{1800}f_n + \frac{11}{90}f_{n+1} + \frac{13}{24}f_{n+2} - \frac{44}{225}f_{n+\frac{5}{2}} + \frac{7}{180}f_{n+3} \right), \tag{15c}$$

$$hz_{n+\frac{5}{2}} = -y_{n+1} + y_{n+2} + h^2 \left(-\frac{19}{3200}f_n + \frac{637}{5760}f_{n+1} + \frac{1537}{1920}f_{n+2} + \frac{17}{225}f_{n+\frac{5}{2}} + \frac{37}{1920}f_{n+3} \right), \tag{15d}$$

$$hz_{n+3} = -y_{n+1} + y_{n+2} + h^2 \left(\frac{7}{900}f_n + \frac{1}{8}f_{n+1} + \frac{7}{10}f_{n+2} + \frac{12}{25}f_{n+\frac{5}{2}} + \frac{73}{360}f_{n+3} \right). \tag{15e}$$

Equations (13) and (14) yields our desired block method which is self-starting for $k = 3$.

Boundary value problem. The following remarks is stated which emphasizes how the method are obtained by imposing that $z_n(x)$ is continuous at $x = x_{n+3}$ as follows

$$\begin{aligned} y_{n+1} \quad - \quad & y_{n+2} - y_{n+4} + y_{n+5} = h^2 (\frac{581}{1800}f_n - \frac{13}{10}f_{n+1} - \frac{19}{40}f_{n+2} - \frac{12}{250}f_{n+\frac{5}{2}} \\ & - \frac{78408}{1800}f_{n+3} \frac{258345}{1800}f_{n+4} - \frac{594900}{1800}f_{n+5} + \frac{567648}{1800}f_{n+\frac{11}{2}} - \frac{160555}{1800}f_{n+6}), \end{aligned} \tag{16}$$

where

$$z_n(x) = \frac{1}{h}(y_{n+1} - y_{n+2} - h^2 \frac{581}{1800}f_n + h^2 \frac{13}{10}f_{n+1} + h^2 \frac{19}{40}f_{n+2} \tag{17a}$$

$$+ h^2 \frac{12}{250}f_{n+\frac{5}{2}} - h^2 \frac{23}{180}f_{n+3}), \quad x_n \le x \le x_{n+3}$$

$$z_n(x) = \frac{1}{h}(y_{n+4} - y_{n+5} - h^2 \frac{78638}{1800}f_{n+3} + h^2 \frac{258345}{1800}f_{n+4} \tag{17b}$$

$$- h^2 \frac{594900}{1800}f_{n+5} + h^2 \frac{567648}{1800}f_{n+\frac{11}{2}} - h^2 \frac{160555}{1800}f_{n+6}), \quad x_{n+3} \le x \le x_{n+6}.$$

2.3. Four-step Hybrid Linear Method with One Off-step Point (4HLM)
We use Equation (11) to obtain a 4-step HLM with the following specification: $r = 3$, $s = 5$, $k = 4$, $v = 7/2$ and $\alpha_j(x)$, $\beta_j(x)$, $\beta_v(x)$ can be expressed as functions of t, given that $t = \frac{x-x_n}{h}$ to obtain the continuous form as follows

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4} + \beta_5 f_{n+7/2}. \tag{18}$$

Initial value problem. Evaluate equation (18), and evaluating at $j = 0, 1, \frac{7}{2}, 4$ yields the following implicit discrete scheme:

$$y_{n+4} = -y_{n+2} + 2y_{n+3} + \frac{1}{560}h^2 f_n - \frac{1}{60}h^2 f_{n+1} + \frac{17}{120}h^2 f_{n+2} \tag{19a}$$

$$+ \frac{41}{60}h^2 f_{n+3} + \frac{16}{105}h^2 f_{n+\frac{7}{2}} + \frac{3}{80}h^2 f_{n+4}.$$

$$y_n = -3y_{n+2} + 2y_{n+3} + \frac{109}{1680}h^2 f_n + \frac{13}{12}h^2 f_{n+1} + \frac{191}{120}h^2 f_{n+2} \tag{20a}$$

$$+ \frac{23}{60}h^2 f_{n+3} - \frac{16}{105}h^2 f_{n+\frac{7}{2}} + \frac{7}{240}h^2 f_{n+4},$$

$$y_{n+1} = 2y_{n+2} - y_{n+3} - \frac{1}{240}h^2 f_n + \frac{1}{10}h^2 f_{n+1} + \frac{97}{120}h^2 f_{n+2} \tag{20b}$$

$$+ \frac{1}{10}h^2 f_{n+3} - \frac{1}{240}h^2 f_{n+4},$$

$$y_{n+\frac{7}{2}} = -\frac{1}{2}y_{n+2} + \frac{3}{2}y_{n+3} + \frac{23}{26880}h^2 f_n - \frac{31}{3840}h^2 f_{n+1} + \frac{269}{3840}h^2 f_{n+2} \tag{20c}$$

$$+ \frac{1279}{3840}h^2 f_{n+3} - \frac{5}{168}h^2 f_{n+\frac{7}{2}} + \frac{17}{1920}h^2 f_{n+4},$$

$$y_{n+4} = -y_{n+2} + 2y_{n+3} + \frac{1}{560}h^2 f_n - \frac{1}{60}h^2 f_{n+1} + \frac{17}{120}h^2 f_{n+2} \tag{20d}$$

$$+ \frac{41}{60}h^2 f_{n+3} + \frac{16}{105}h^2 f_{n+\frac{7}{2}} + \frac{3}{80}h^2 f_{n+4}.$$

Differentiating the continuous scheme with respect to x and evaluating at x_n, x_{n+1}, x_{n+2}, x_{n+3}, $x_{n+7/2}$, and x_{n+4} yields the following discrete scheme:

$$hz_n = -y_{n+2} + y_{n+3} + h^2(-\frac{4357}{14112}f_n - \frac{3637}{2520}f_{n+1} - \frac{1891}{5040}f_{n+2} - \frac{1987}{2520}f_{n+3} \tag{21a}$$

$$+ \frac{1192}{2205}f_{n+\frac{7}{2}} - \frac{1987}{2520}f_{n+4}),$$

$$hz_{n+1} = -y_{n+2} + y_{n+3} + h^2(\frac{923}{70560}f_n - \frac{499}{1260}f_{n+1} - \frac{5629}{5040}f_{n+2} \tag{21b}$$

$$+ \frac{137}{1260}f_{n+3} - \frac{64}{441}f_{n+\frac{7}{2}} + \frac{73}{2016}f_{n+4}),$$

$$hz_{n+2} = -y_{n+2} + y_{n+3} + h^2(-\frac{169}{70560}f_n - \frac{59}{2520}f_{n+1} - \frac{1667}{5040}f_{n+2} \tag{21c}$$

$$- \frac{151}{504}f_{n+3} - \frac{296}{2205}f_{n+\frac{7}{2}} + \frac{251}{10080}f_{n+4}),$$

$$hz_{n+3} = -y_{n+2} + y_{n+3} + h^2(\frac{139}{70560}f_n - \frac{23}{1260}f_{n+1} + \frac{151}{1008}f_{n+2} \tag{21d}$$

$$+ \frac{613}{1260}f_{n+3} - \frac{64}{441}f_{n+\frac{7}{2}} + \frac{253}{10080}f_{n+4}),$$

$$hz_{n+\frac{7}{2}} = -y_{n+2} + y_{n+3} + h^2(\frac{1643}{1128960}f_n - \frac{71}{5040}f_{n+1} + \frac{10561}{80640}f_{n+2} \tag{21e}$$

$$+ \frac{15317}{20160}f_{n+3} + \frac{199}{1764}f_{n+\frac{7}{2}} + \frac{293}{32256}f_{n+4}),$$

$$hz_{n+4} = -y_{n+2} + y_{n+3} + h^2(\frac{167}{70560}f_n - \frac{53}{2520}f_{n+1} + \frac{797}{5040}f_{n+2} \tag{21f}$$

$$+ \frac{1597}{2520}f_{n+3} + \frac{1192}{2205}f_{n+\frac{7}{2}} + \frac{1877}{10080}f_{n+4}).$$

Equations (20)-(21) yields our desired block method which is self-starting for $k = 4$.

Boundary value problems. The following remarks is stated which emphasizes how the

method are obtained by imposing that $z_n(x)$ is continuous at $x = x_{n+4}$ as follows

$$
\begin{aligned}
-y_{n+2} \; + \; & y_{n+3} - y_{n+6} + y_{n+7} = h^2 \big(\frac{4357}{14112} f_n + \frac{3637}{2520} f_{n+1} + \frac{1891}{5040} f_{n+2} + \frac{1987}{2520} f_{n+3} \\
& - \; \frac{1192}{2205} f_{n+\frac{7}{2}} - \frac{11335944}{2520} f_{n+4} + \frac{47163956}{70560} f_{n+5} \\
& - \; \frac{94771306}{70560} f_{n+6} + \frac{158400172}{70560} f_{n+7} - \frac{133917440}{70560} f_{n+\frac{15}{2}} \frac{34057555}{70560} f_{n+8} \big)
\end{aligned}
\tag{22}
$$

where

$$
z_n(x) = \frac{1}{h} \big(-y_{n+2} + y_{n+3} - h^2 \frac{4357}{14112} f_n - h^2 \frac{3637}{2520} f_{n+1} - h^2 \frac{1891}{5040} f_{n+2}
\tag{23a}
$$

$$
- \; h^2 \frac{1987}{2520} f_{n+3} + h^2 \frac{1192}{2205} f_{n+\frac{7}{2}} - h^2 \frac{1987}{2520} f_{n+4} \big), \quad x_n \le x \le x_{n+4}
$$

$$
z_n(x) = \frac{1}{h} \big(y_{n+6} - y_{n+7} - h^2 \frac{11391577}{70560} f_{n+4} + h^2 \frac{47163956}{70560} f_{n+5} - h^2 \frac{94771306}{70560} f_{n+6}
\tag{23b}
$$

$$
+ \; h^2 \frac{158400172}{70560} f_{n+7} - h^2 \frac{133917440}{70560} f_{n+\frac{15}{2}} + h^2 \frac{34057555}{70560} f_{n+8} \big), \quad x_{n+4} \le x \le x_{n+8}.
$$

3. Order and Error Constant of Methods

We define a local truncation error associated with a second order differential equation by the difference operator

$$
L[y(x); h] = \sum_{i=0}^{k} [\alpha_i y(x_n + ih) - h^2 \beta_i f(x_n + ih)],
\tag{24}
$$

here $y(x)$ is an arbitrary function, continuously differentiable on the interval $[a, b]$. Expanding the expression (24) in the Taylors series about the point x, we obtain:

$$
L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_{p+2} h^{p+2} y^{(p+2)}(x)
\tag{25}
$$

where vectors

$$
C_0 \; = \; \sum_{i=0}^{k} \alpha_i, \quad C_1 = \sum_{i=0}^{k} i \alpha_i, \quad C_2 = \frac{1}{2!} \sum_{i=0}^{k} i \alpha_i - \beta_i, \ldots,
$$

$$
C_p \; = \; \frac{1}{p!} \sum_{i=0}^{k} i^p \alpha_i - p(p-1)(p-2) i^{p-2} \beta_i, \quad \text{where} \quad p = 3, 4, 5 \ldots .
$$

According to Lambert [15] the method's order is p if $C_0 = C_1 = C_2 = \ldots = C_p = C_{p+1} = 0$ and $C_{p+2} \ne 0$. Therefore, C_{p+2} is the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ is the principal local truncation error at the point x_n. From this definition, we obtain the order of 3SHLM (14)-(15), 4SHLM (20)-(21) are 5 and 6, respectively. The corresponding error constants are given in Table 1.

3.1. Zero-stability of the Methods

To analyze the zero-stability of the methods the characteristic polynomials $\rho(z)$ of proposed 3SHLM (14), 4SHLM (20) are shown in Table 2. Equating $\rho(z)$ to zero and solving for z gives real roots $\{z_i\}_{i=1}^{k}$, whose orders of multiplicity are $\{m_i\}_{i=1}^{k}$, respectively, and let $\{r_j, \bar{r}_j\}_{i=1}^{p}$ be distinct pairs of complex conjugate solutions of multiplicity $\{\mu_i\}_{i=1}^{p}$ polynomial.

Table 1. Order and Error Constant

Steps	Equations	Order	Error constant, C_{p+2}
$k = 3$	(14)	$p = 5$	$\left[-\frac{1}{800}, -\frac{23}{38400}, -\frac{7}{2400}\right]^{\top}$
	(15)		$\left[\frac{923}{56}, \frac{169}{56}, \frac{139}{56}, \frac{1643}{56}, \frac{167}{56}\right]^{\top}$
$k = 4$	(20)	$p = 6$	$\left[-\frac{1}{1890}, -\frac{31}{60480}, -\frac{1}{630}, -\frac{2531}{10321920}\right]^{\top}$
	(21)		$\left[-\frac{1624}{3}, -\frac{539}{4}, -\frac{140}{3}, -\frac{10619}{24}, -\frac{497}{12}, -56\right]^{\top}$

Table 2. Characteristic polynomials, their roots and orders of multiplicity

Steps	Equations	Polynomials, $\rho(z)$	Roots, $\{z_i\}$ and Order of multiplicity, $\{m_i\}$ and $\{\mu_i\}$
$k = 3$	(14a)	$z^2 - 2z + 1$	$z_1 = 1, m_1 = 2$
	(14b)	$z^{\frac{5}{2}} - \frac{3}{2}z^2 + \frac{1}{2}z$	$z_1 = 0, z_2 = 1$
	(14c)	$z^3 - 2z^2 + z$	$z_1 = 0, m_1 = 2; z_2 = 1, m_2 = 1$
$k = 4$	(20b)	$2z^3 - 3z^2 + 1$	$z_1 = 1, m_1 = 2; z_2 = -\frac{1}{2}, m_2 = 1$
	(20c)	$z^3 - 2z^2 + z$	$z_1 = 0, m_1 = 2; z_2 = 1, m_2 = 1$
	(20d)	$z^{\frac{7}{2}} - \frac{3}{2}z^3 + \frac{1}{2}z^2$	$z_1 = 0, z_2 = 1$
	(21a)	$z^4 - 2z^3 + z^2$	$z_1 = 0, m_1 = 2; z_2 = 1, m_2 = 2$

According to Lambert [15], a LMM is said to be *zero-stable* if no root of its characteristic polynomial $\rho(z)$ has no modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

As one can see from Table 2 the characteristic polynomials $\rho(z)$ are fractional order polynomials for thee off-step point cases (14b), (20d), respectively, with the highest powers are $\{5/2, 7/2, 9/2\} \in \mathbb{R}$. These polynomials have two different real roots: 0 and 1, whose the orders of multiplicity are less than the orders of the methods p (Table 1), respectively. We can conclude that in all cases the proposed methods are zero-stable.

3.2. Consistency and Convergence of the Methods

According to Lambert [15], a linear multistep method is said to be *consistent* if it has order at least one, $p > 1$. Owing to this definition, proposed methods described by equations (14)-(15), (20)-(21) are consistent.

According to the equivalence theorem of Dahlquist, the necessary and sufficient condition for a LMM to be *convergent* is to be consistent and zero-stable. Since the two conditions are satisfied, it follows the convergence of methods.

4. Region of Absolute Stability of the Methods

Stability regions are a standard tool in the analysis of numerical formulas for ODE problems. To evaluate and plot the region of absolute stability of HLM, the methods were reformulated as general linear method expressed as [16]:

$$\left[\begin{array}{c} Y \\ y_{i+1} \end{array}\right] = \left[\begin{array}{cc} A & U \\ B & V \end{array}\right] \left[\begin{array}{c} hF(Y) \\ y_{i+1} \end{array}\right], \tag{26}$$

where

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{array}\right], \quad B = \left[\begin{array}{cccc} b_{11} & b_{12} & \ldots & b_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{s1} & b_{s2} & \cdots & b_{ss} \end{array}\right],$$
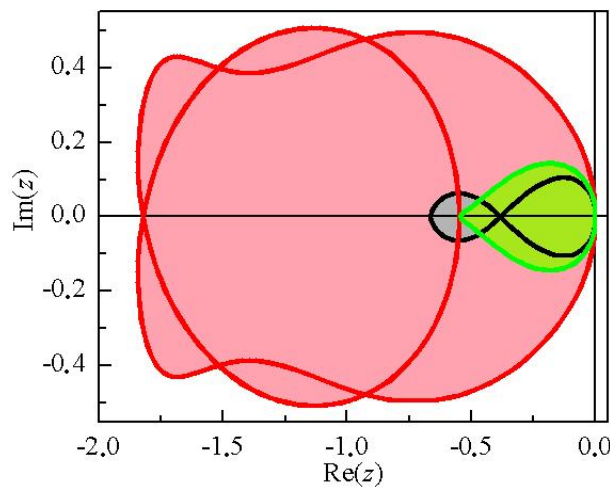
Figure 1. Stability regions of the hybrid linear methods with one off-step point in the complex plan: 3SHLM – red line, 4SHLM – black line, 5SHLM – green line

$Y = (y_n, y_{n+1}, y_{n+2})^\top$, $y_{i+1} = (y_{n+k}, y_{n+k-1})^\top$, $y_{i-1} = (y_{n+k-1}, y_{n+k-2})^\top$. Also the elements of the matrices A, B, U and V were obtained from interpolation and collocation points and then substituted into the stability matrix as

$$M(z) = V + zB(I - zA)^{-1}U, \quad z \in \mathbb{C}, \tag{27}$$

here I is identity matrix, and the stability matrix $M(z)$ (27) was substituted into the stability function

$$\rho(\eta, z) = \det(\eta I - M(z)), \tag{28}$$

and then computed with Maple Software to yield the stability polynomial $f(z)$.

Case of 3SHLM. Translating the coefficients of Equations (14)-(15) into the (A, U, B, V) formulation gives the matrix:

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{3}{74} & \frac{75}{74} & 0 & \frac{903}{1850} \\ 0 & 0 & -\frac{28}{129} & 0 & \frac{1}{129} & 0 & \frac{50}{129} & -\frac{7}{215} \\ 0 & 0 & 0 & \frac{35}{76} & -\frac{115}{608} & \frac{975}{608} & -\frac{215}{304} & \frac{63}{608} \\ 0 & 0 & 0 & 0 & \frac{2}{28} & -\frac{51}{28} & \frac{5}{14} & -\frac{1}{20} \\ \hline 0 & 0 & 0 & 0 & \frac{2}{28} & -\frac{51}{28} & \frac{5}{14} & -\frac{1}{20} \\ 0 & 0 & -\frac{28}{129} & 0 & \frac{1}{129} & 0 & \frac{50}{129} & -\frac{7}{215} \\ 0 & 0 & 0 & 0 & \frac{3}{74} & \frac{75}{74} & 0 & \frac{903}{1850} \end{array}\right]. \tag{29}$$

By substituting the entries of the matrices (29) into the Equations (27)-(28), the stability polynomial of 3SHLM is

$$f(z) = \frac{1}{925} \frac{1}{(3z - 28)(28z + 129)} \left(\begin{array}{c} 77700\eta^3 z^2 - 39396\eta^2 z^2 + 672252\eta z \\ -1277927\eta^2 z - 367225\eta^3 z - 3159757\eta^2 \\ +4527756\eta - 3341100\eta^3 - 1125075 \end{array}\right).$$

Case of 4SHLM. We obtained the explicit form the (A, U, B, V) by analogy with the 3SHLM case. By substituting the entries of the matrix (A, U, B, V) into the Equations (27)-(28), the

stability polynomial of 4SHLM is

$$f(z) = \frac{\left(\begin{array}{l} -2937687750\eta^3 z^2 - 6558375600\eta^4 z^3 - 23047427340\eta^4 z^2 \\ -174998144925\eta^3 z^2 + 79233752475\eta^2 z^2 + 1035308237580\eta^2 z - 857606540895\eta^3 z \\ -2429959455545\eta z - 483980940156\eta^4 z + 223739742762\eta^2 - 77747635862\eta^3 \\ +198449239366 - 976274034120\eta^4 - 1194320318598\eta \end{array}\right)}{9996\left(1620z^2 - 14089z - 46530\right)\left(405z + 2099\right)}.$$

The regions of absolute stability for the proposed methods are shown in Figure 1. It was found that the interval of absolute stability for the 3SHLM is $(-0.6630, 0)$, 4SHLM is $(-1.82, 0)$, 5SHLM is $(-0.5445, 0)$. The stability region contains the entire left half complex plane and thus, the method is A-stable.

5. Numerical Examples
In this section, the results of the proposed method developed in section 4 are presented for some IVP and BVP of second order differential equations.

We consider four numerical examples: the Van Der Pol Oscillator Problem [7], the IVP of Bratu-type [8], the Troesch's Problem [9, 10, 11] to test the efficiency of the derived orthogonal-based hybrid block method, $k = 3, 4, 5$.

Problem 1. Van Der Pol Oscillator [7]

$$y'' - 2\xi\left(1 - y^2\right)y' + y = 0, \quad y(0) = 0, \quad y'(0) = 0.5, \quad x \in [0, 10], \quad \xi = 0.025.$$

The solutions comparison of proposed methods with the Runge-Kutta (RK45) method is given in Fig. 2 a.
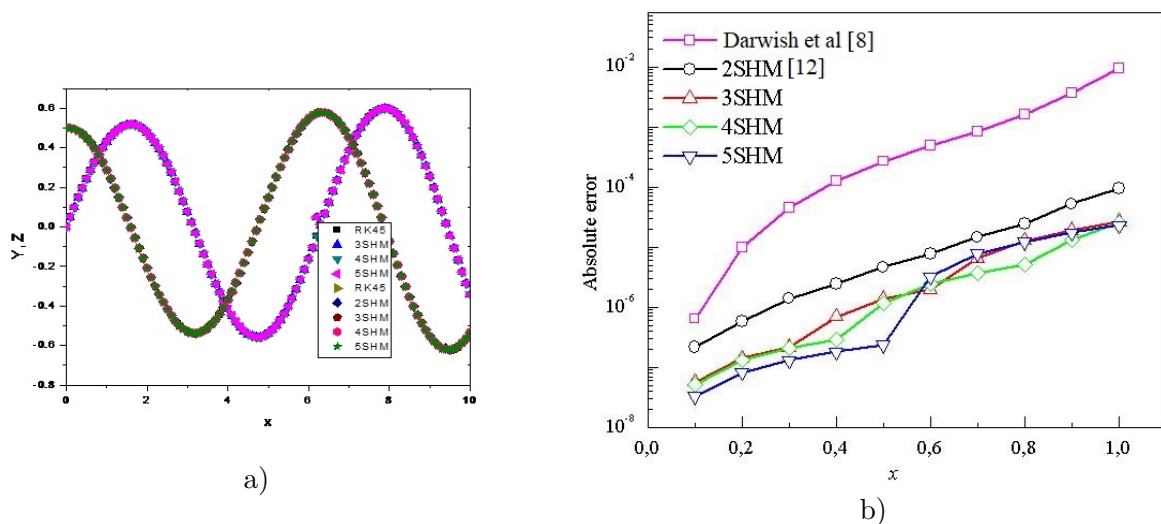

Figure 2. a) Solutions comparison of proposed methods with the Runge-Kutta (RK45) method for Problem 1 [7], b) Comparison absolute error of proposed methods with the methods Darwish et al [8] and 2SHM [12] for Problem 2 [8].

Problem 2. Consider the second order initial value problem of Bratu type [8]

$$y'' - 2\exp(y) = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad 0 \le x \le 1.$$

The exact solution is $y(x) = -2\ln(\cos x)$. The comparison absolute error $|y(x) - y(x_i)|$, $i = 1, 2, \ldots, 10$, $h = 0.1$ of proposed methods with the methods [8, 12] is given in Fig. 2 b.

Table 3. Comparison of Solutions for Proposed Methods for Troesch's Problem

x	3SHM	4SHM	Khuri [9]	Hassan [17]	Deeba [18]
0.0	0	0	0	0	0
0.1	0.096038851	0.096038616	0.095944352	0.095945	0.095938
0.2	0.192317712	0.192317240	0.192128754	0.192129	0.192118
0.3	0.289076632	0.289075922	0.288794411	0.288795	0.288780
0.4	0.386555574	0.386554788	0.386184861	0.386186	0.386169
0.5	0.484994905	0.484994083	0.484547183	0.484549	0.484530
0.6	0.584634871	0.584634013	0.584133265	0.584135	0.584117
0.7	0.685715495	0.685714786	0.685201168	0.685203	0.685187
0.8	0.788476985	0.788476423	0.788016546	0.788018	0.788006
0.9	0.893158939	0.893158553	0.892854236	0.892855	0.892848
1.0	1	1	1	1	1

Problem 3. Consider the Troesch's Problem [11]

$$y'' = n \sinh(ny), \quad y(0) = 0, \quad y(1) = 1, \quad 0 \leq x \leq 1.$$

Table 3 shown the comparison between approximate solutions obtained with the proposed methods and methods from literature, using the Troesch's parameter $n = 0.5$.

Conclusion

A collocation technique which yields a method with continuous coefficients has been presented for the approximate solution of second order ODEs.

In this work, we obtained an approximate solution for different second order initial and boundary value problems: the Van der Pol Oscillator problem, the Bratu's type problem, the Troeschs problem. Besides, we presented a comparison between the numerical solution, the proposed solution, and other approximations reported in the literature.

Results of this research can be used as the foundation for methods, which solve initial and boundary value problems for second order equations. The programs have been provided by Maple Software.

References
[1] D'Ambrosio R and Ferro M 2009 *Applied Mathematics Letters* **22** 1076–1080
[2] Enright W H 1974 *SIAM. J. Numer. Anal.* **11** 321–331
[3] Hairer E and Wanner G 2002 *Solving Ordinary Differential Equations II Stiff and Differential Algebraic Problems* (Berlin: Springer)
[4] Anake T A 2012 *International Journal of Applied Math.* **42** 224–228
[5] Okuonghae R I and Ikhile M N O 2014 *Siberian J. Num. Math.* **17** 67–81
[6] Jator S N and Oladejo H B 2017 *International J. of Applied and Computational Mathematics* **3** 1385–1402
[7] Majid Z A 2012 *Math. Problems in Engineering*
[8] Darwish M A and Kashkari B S 2014 *American Journal of Computational Mathematics* **4** 47–54
[9] Khuri S A 2003 *International Journal of Computer Mathematics* **80** 493–498
[10] Mirmoradi S H, Hosseinpour I, Ghanbarpour S and Barari A 2009 *Applied Mathematical Sciences* **3** 1579–158
[11] Vazquez-Leal H 2012 *Mathematical Problems in Engineering* **2012**
[12] Mohammed U, Oyelami O and Semenov M 2019 *J. Phys.: Conf. Ser.* **1145** 1–8
[13] Fischer B and Golub G H 1992 *Journal of Computational and Applied Mathematics* **43** 99–115
[14] Gautschi W 2004 *Orthogonal polynomials: computation and approximation* (Oxford: Oxford University Press)
[15] Lambert J D 1973 *Computational method in ordinary differential equation* (London: John Wiley and Sons)
[16] Butcher J C 2016 *Numerical methods for ordinary differential equations* (London: John Wiley and Sons)
[17] Hassan H N and El-Tawil M A 2011 *Mathematical Methods in the Applied Sciences* **34** 977–989
[18] Deeba E, Khuri S A and Xie S 2000 *Journal of Computational Physics* **159** 125–138