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Direct integrators of modified multistep method for the solution of third order boundary value problem in ordinary differential equations

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Abstract. In this paper, we propose an efficient modified multistep method for direct solution of boundary value problems (BVPs) using multistep collocation approach. The continuous form was evaluated at grid and off-grid points to obtain the multiple finite difference schemes. The basic properties, such as order and error constants, zero stability and convergence analysis of the proposed methods were investigated. Numerical experiment were performed to show the efficiency of the method and the results were compared with the existing methods in the literature.

1. Introduction

The application of mathematical formulation in science and engineering is given by a boundary value problem (BVP):

$$y''' = f(x, y, y', y''), \quad y(a) = y_0, \quad y'(a) = \delta_0, \quad y(b) = y_M, \tag{1a}$$

$$y''' = f(x, y, y', y''), \quad y(a) = y_0, \quad y'(a) = \delta_0, \quad y'(b) = y_M. \tag{1b}$$

There are many methods to solve third order ordinary differential equations (ODEs) (1). Most of these methods are solved BVP by reducing a higher ODEs to an equivalent system of first order ODEs which take a lot of time and human effort. Alternative approach is to solve higher ODEs directly. In the paper [1] authors investigated two- and three-stage Runge-Kutta type methods for special third order ODEs. Higher order linear multi-step methods were proposed by Jator [2] to the numerical integration of third order BVP. The case of the four-points block hybrid collocation method with two off-step points to solve general third order ODEs directly was studied by Yap and Ismail [3]. In the paper by Jikantoro et al. [4] presented the theory of B-series and the associated rooted trees through which order conditions of the hybrid methods for direct integration of special third order ODEs are derived.

In this research, we develop a continuous hybrid liner multistep method (HLMM) for direct solution of BVPs without reducing the problem to a lower order system or to an IVP equivalent. The proposed HLMM is zero stable, consistent and more accurate than the existing one. Experimental results confirm the superiority of the new schemes over the existing methods.

2. Development of the Method

Collocation solutions are desirable from practical and theoretical considerations and their advantages are now creating growing interest in continuous integration algorithms for numerical solution of ODEs. In particular, collocation solutions of the ODEs by their nature are continuous.

In the spirit of Onumanyi et al. [5] and Mohammed [6] we consider briefly the derivation of the continuous formula by the multistep collocation using constant mesh spacing h and give explicit representation for the coefficients.

The values of k and m are arbitrary except for collocation at the mesh points, where $0 < m \leq k+1$.

Let y_{n+j} be approximations to $y(x_{n+j})$ where $y_{n+j} = y(x_{n+j})$, $j = 0, 1, \ldots, k$, then a k-step multistep collocation formula with m collocation points is constructed as follows:

$$y(x_{n+k}) = \sum_{j=0}^{r-1} \bar{\alpha}_j y_{n+j} = h^3 \sum_{j=0}^{m-1} \bar{\beta}_j f(\bar{x}_j, y(\bar{x}_j)) + h^3 \beta_\eta f(x_\eta, y(x_\eta))$$

where $\bar{\alpha}_j$, $j = 0, 1, \ldots, r-1$, $\bar{\beta}_j$, $j = 0, 1, \ldots, m-1$, and β_η, $\eta \in \mathbb{R}$ are unknown constants of the discrete scheme. To obtain a continuous form of Eq. (2) we find the polynomial $y(x)$ of degree $p = r + m - 1$, $r > 0$, $m > 0$ of the form

$$y(x) = \sum_{j=0}^{r-1} \bar{\alpha}_j(x) y_{n+j} + h^3 \sum_{j=0}^{m-1} \bar{\beta}_j(x) f(\bar{x}_j, y(\bar{x}_j)) + h^3 \beta_\eta(x) f(x_\eta, y(x_\eta)), \tag{2}$$

such that it satisfies the conditions

$$\begin{aligned} \bar{\alpha}_j y(x_{n+j}) &= \bar{\alpha}_j y_{n+j}, & j = 0, 1, \ldots, r-1, \\ \bar{\beta}_j y'''(x_j) &= \bar{\beta}_j f(\bar{x}_j, y(\bar{x}_j)), & j = 0, 1, \ldots, m-1, \end{aligned}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are assumed polynomials of the form

$$\alpha_j(x) = \sum_{i=0}^{r+m-1} \alpha_{j,i+1} x^i, \quad h^3 \beta_j(x) = \sum_{i=0}^{r+m-1} \beta_{j,i+1} x^i.$$

Points x_{n+j} in Eq. (2) are r arbitrarily chosen *interpolation* points taken from the range $\{x_n, \ldots, x_{n+k-1}\}$, $0 < r \leq k$, and the *collocation* points $\bar{x}_j$, $j = 0, 1, \ldots, m-1$ belongs to the extended set $Q = \{x_n, \ldots, x_{n+k}\} \bigcup \{x_{n+k-1}, x_{n+k}\}$.

From the interpolation conditions and the expression for $y(x)$ in Eq. (2) the following conditions are imposed on continuous coefficients $\alpha_j(x)$ and $\beta_j(x)$:

$$\begin{aligned} \alpha_j(x_{m+i}) &= \delta_j^i, \quad j = 0, 1, \ldots, r-1, \quad i = 0, 1, \ldots, r-1 \\ h^3 \beta_j(x_{n+i}) &= 0, \quad j = 0, 1, \ldots, m-1, \quad i = 0, 1, \ldots, r-1 \end{aligned}$$

and

$$\begin{aligned} \alpha_j'''(\bar{x}_{n+i}) &= 0, \quad j = 0, 1, \ldots, r-1, \quad i = 0, 1, \ldots, m-1 \\ h^3 \beta_j(x_{n+i}) &= \delta_j^i, \quad j = 0, 1, \ldots, m-1, \quad i = 0, 1, \ldots, m-1, \end{aligned}$$

here $\delta_j^i = [i = j]$ is the Kronecker delta.

We note that since the general third order ODEs (1) involves the first and second derivatives, the first and second derivative formulas are

$$y'(x) = \frac{1}{h}\left(\sum_{j=0}^{r-1}\alpha_j'(x)y_{n+j} + h^3\sum_{j=0}^{k}\beta_j'(x)y_{n+j} + h^3\beta_\eta'(x)f_{n+\eta}\right),$$

$$y''(x) = \frac{1}{h^2}\left(\sum_{j=0}^{r-1}\alpha_j''(x)y_{n+j} + h^3\sum_{j=0}^{k}\beta_j''(x)y_{n+j} + h^3\beta_\eta''(x)f_{n+\eta}\right),$$

and to obtain additional equations by imposing that

$$y'(x) = \delta(x), \quad y''(x) = \gamma(x), \quad y'(a) = \delta_0, \quad y''(a) = \gamma_0.$$

The method of continuous approximation can be expressed as

$$y(x) = \sum_{j=0}^{r-1}\alpha_j(x)y_{n+j} + h^3\sum_{j=0}^{k}\beta_j(x)y_{n+j} + h^3\beta_\eta(x)f_{n+\eta}. \tag{3}$$

3. Three-step hybrid method with one off-step collocation point

We use Equation (3) to obtain a three-step HLMM with the following specification: $r = 3$, $m = 5$, $\eta = \frac{8}{3}$, $k = 3$, $\alpha_j(x)$, $\beta_j(x)$, $\beta_\eta(x)$ can be expressed as functions of $t = \frac{x-x_n}{h}$. The HLMMs are usually represented in the form of a single block r-point multistep method [7]

$$\mathbf{A}Y_m = \mathbf{B}Y_{m-1} + h^3\mathbf{C}F_m + h^3\mathbf{D}F_{m-1} \tag{4}$$

where h a fixed mesh size within a block hybrid, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D} \in \mathbb{R}^{(k+1)\times(k+1)}$ are the coefficient matrices, Y_m, Y_{m-1}, F_m and $F_{m-1} \in \mathbb{R}^{k+1}$ are vectors of numerical approximation.

The hybrid method can be significantly shown in the form of Equation (4) to give

$$\begin{pmatrix} 3 & -3 & 0 & 1 \\ \frac{16}{9} & -\frac{20}{9} & 1 & 0 \\ -2 & \frac{1}{2} & 0 & 0 \\ 2 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{8}{3}} \\ y_{n+3}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$
$$+ h^3\begin{pmatrix} \frac{47}{100} & \frac{23}{40} & -\frac{81}{800} & \frac{1}{20} \\ \frac{7}{27} & \frac{203}{729} & -\frac{25}{324} & \frac{65}{2187} \\ \frac{1399}{4200} & -\frac{23}{168} & \frac{783}{5600} & -\frac{17}{2800} \\ -\frac{109}{120} & \frac{61}{120} & -\frac{81}{160} & \frac{79}{360} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{8}{3}} \\ f_{n+3}\end{pmatrix} + h^3\begin{pmatrix} 0 & 0 & 0 & \frac{1}{160} \\ 0 & 0 & 0 & \frac{31}{8748} \\ 0 & 0 & 0 & -\frac{13}{224} \\ 0 & 0 & 0 & -\frac{451}{1440} \end{pmatrix}\begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}. \tag{5}$$

4. Order and Error Constant of Hybrid Linear Multi-step Method

With specific reference to the works of Fatunla [7] and Lambert [8], the local truncation error attributed to the conventional form of Equation (3) is defined as the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k}\left\{\alpha_j y(x+jh) - h^3\beta_j y'''(x+jh)\right\} - h^3\beta_\eta y'''(x+\eta h). \tag{6}$$

Suppose it is assumed that $y(x)$ can be adequately differentiated. It is possible to expand Equation (6) in the form of Taylor series about the point x to arrive at the expression

$$L[y(x);h] = C_0 y(x) + C_1 y'(x) + \ldots + C_q h^{(q)} y^{(q)}(x) + \ldots$$

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where the constant coefficients C_q, $q = 0, 1, \ldots$ are given as shown below:

$$C_0 = \sum_{j=0}^{k} \alpha_j, \quad C_1 = \sum_{j=1}^{k} j\alpha_j, \ldots, C_q = \frac{1}{q!} \sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2) \left(\sum_{j=1}^{k} j^{q-3} \beta_j + \eta^{q-3} \beta_\eta \right).$$

According to the paper by Henrici [9], the method (2) has the order p if

$$C_0 = C_1 = \ldots = C_p = C_{p+1} = 0, \quad C_{p+2} = 0, \quad \text{and} \quad C_{p+3} \neq 0.$$

Therefore, C_{p+3} is the error constant. The proposed method (5) has a uniform order $p = 5$ and the error constants as

$$C_{p+3} = \left(-\frac{7}{7200}, -\frac{85}{157464}, -\frac{811}{18}, -\frac{197}{14} \right)^\top.$$

In order to analyze the method (5) for zero stability, we normalize the scheme and write it as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(R \cdot \mathbf{A}^{(0)} - \mathbf{A}^{(1)}) = R^k(R - 1).$$

It is easily shown that method (5) is normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(R \cdot \mathbf{A}^{(0)} - \mathbf{A}^{(1)}) = R^3(R - 1)$$

where $\mathbf{A}^{(0)} = \mathbf{1}_{4\times 4}$ is the identity matrix of dimension 4, $\mathbf{A}^{(1)} = \mathbf{1}_{4\times 1} \cdot \mathbf{i}_{4,4}^{\mathrm{T}}$ is the matrix of dimension 4, $\mathbf{i}_{4,4}$ is the 4-th column of $A^{(0)}$, and roots of the characteristic polynomial $\rho(R)$ are $R = (0, 0, 0, 1)$. Therefore, the proposed method is zero-stable.

5. Region of Absolute Stability

The boundary locus method is used to obtain the stability region of the main method of Equation (5). The boundary locus curve is obtained by setting

$$\bar{h} = \lambda h = \frac{\rho(z)}{\sigma(z)}, \quad \lambda > 0, \quad z = \exp\{i \cdot \theta\} \in \mathbf{C}, \quad \theta \in [0, 2\pi], \tag{7}$$

where $\rho(z)$, $\sigma(z)$ are first and second characteristic polynomial of liner multistep method. The main method from Equation (5) is written as

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = h^3 \left(\frac{1}{160} f_n + \frac{47}{100} f_{n+1} + \frac{23}{40} f_{n+2} - \frac{81}{800} f_{n+\frac{8}{3}} + \frac{1}{20} f_{n+3} \right). \tag{8}$$

The first and second characteristics polynomials are written as

$$\rho(z) = z^3 - 3z^2 + 3z - 1, \quad \sigma(z) = \frac{1}{160} + \frac{47}{100} z + \frac{23}{40} z^2 - \frac{81}{800} z^{\frac{8}{3}} + \frac{1}{20} z^3.$$

Substituting $\rho(z)$ and $\sigma(z)$ into the Equation (7), the values of $\bar{h}$ is obtain which is plotted to produce the region of absolute stability of the method (Figure 1). The stability region for the method (8) turns out to be the inside part of the complex plane shown in Figure 1, including the boundary colored by green [11]. From the Figure 1 we found that the interval of absolute stability is $(-1.5, 0)$.

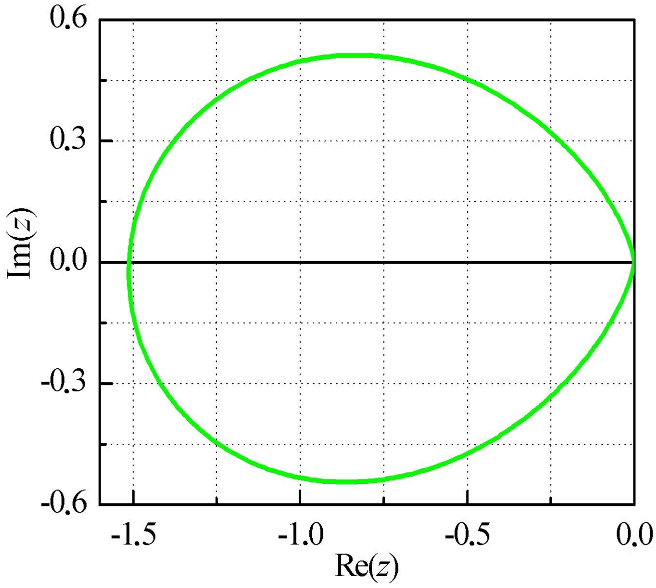

Figure 1. Region of absolute stability for the proposed method

Table 1. Numerical Results for Problem 2

x	Exact Solution	Numerical Solution	Error	Tirmizi et. al [12]
0.0	-0.01210709	-0.012107056410	3.36×10^{-8}	6.6530×10^{-5}
0.1	-0.01126851	-0.011268451800	5.82×10^{-8}	6.5000×10^{-5}
0.2	-0.00922221	-0.009222146339	6.37×10^{-8}	5.2254×10^{-5}
0.3	-0.00646687	-0.006466811798	5.82×10^{-8}	3.6300×10^{-5}
0.4	-0.00332019	-0.003320153971	3.60×10^{-8}	1.8750×10^{-5}
0.6	0.00332019	0.003320118521	7.15×10^{-8}	1.7340×10^{-5}
0.7	0.00646687	0.006466679335	1.91×10^{-7}	3.4050×10^{-5}
0.8	0.00922221	0.009221906116	3.04×10^{-7}	4.9801×10^{-5}
0.9	0.01126851	0.011268057530	4.52×10^{-7}	6.2020×10^{-5}
1.0	0.01210709	0.012106500950	5.89×10^{-7}	6.3480×10^{-5}

6. Numerical Experiments

The accuracy of the proposed method was implemented for direct solution of BVPs of third order ODEs of linear and non-linear equations. The implementation of the method was coded using the Maple Software.

Problems 1. Non-linear Blasius Equation [10]

$$2y''' + yy'' = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y'(\infty) = 1.$$

The exact solution does not exist. Comparison numerical solutions using the proposed method and the fourth order Runge-Kutta (RK) method for the Problem 1 is shown in Fig. 2.

Problems 2. Sandwich Beam Problem [12]

$$y''' - k^2y' + r = 0, \quad y(0.5) = 0, \quad y'(0) = 0, \quad y'(1) = 0.$$

The proposed schemes for the values $k = 1$ and $r = 5$ are relatively more accurate than the schemes of Tirmizi et al. [12] for Problem 2. Absolute errors are presented in Table 1.

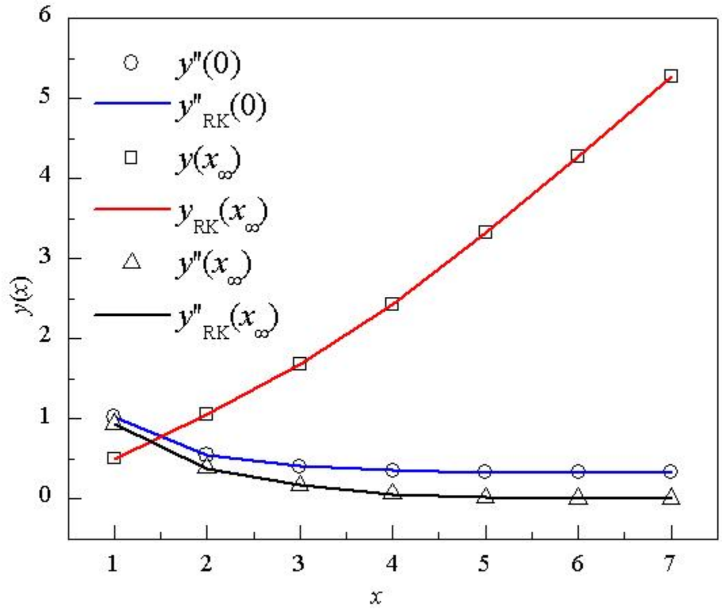

Figure 2. Comparison numerical solution of proposed method with Runge-Kutta method for the Problem 1

7. Conclusion

This research describes the development, analysis and implementation of block methods for solving third order ordinary differential equations directly. The development and/or construction of class of hybrid linear multi-step methods for direct solution of initial value problems and boundary value problems arising from third order ODEs have been presented. The derived schemes which are of block form were analyzed and applied to some selected and standard problems from literature.

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