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A Class of Implicit Six Step Hybrid Backward Differentiation Formulas for the Solution of Second Order Differential Equations

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Abstract

In this paper, we propose a class implicit six step Hybrid Backward Differentiation Formulas (HBDF) for the solution of second order Initial Value Problems (IVPs). The method is derived by the interpolation and collocation of the assumed approximate solution. The single continuous formulation derived is evaluated at grid point of $x = x_{n+k}$ and its second derivative at $x = x_{n+j}$, $j = 1,2,.....k-1$ and $x = x_{n+\mu}$ respectively, where k is the step number of the methods. The interpolation and collocation procedures lead to a system of (k+1) equations, which are solved to determine the unknown coefficients. The resulting coefficients are used to construct the approximate continuous solution from which the Multiple Finite Difference Methods (MFDMs) are obtained and simultaneously applied to provide the direct solution to IVPs. Numerical examples are given to show the efficiency of the method.

Keywords: Hybrid method, backward differentiation formulas, collocation, interpolation, second order, multiple finite differences.

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1 Introduction

In recent times, the integration of Ordinary Differential Equations (ODEs) are investigated using some kind of block methods. This paper discusses the family of implicit Linear Multistep Method (LMM) for numerical integration of general second order ODEs which arise frequently in the area of science and engineering especially mechanical system, control theory and celestial mechanics and are generally written as:

$$y'' = f(x, y', y), \;\; y(a) = y_0 \;,\; y'(a) = \eta_0 \tag{1}$$

In practice the problems are reduced to systems of first order equations and any method for first order equations is used to solve them see Awoyomi [1]. It has been extensively discussed that due to the dimension of the problem after it has been reduced to a system of first order equations also, more often the reduced systems of ordinary differential equations (ODEs) is not well posed, unlike the given problem. The approach waste a lot of Computer time and human efforts, hence there is a need to develop algorithms to handle these classes of problems directly without any reduction to system of first order ODEs.

Development of LMM for solving ODE can be generated using methods such as taylor's series, numerical interpolation, numerical integration and collocation method, which are restricted by an assumed order of convergence. In this paper we will consider the contribution of multi step collocation technique introduced by Onumayi et al. [2] by deriving our new method. Some researchers have attempted the solution of directly using linear multistep methods without reduction to system of first order ordinary differential equations the include Mohammed et al. [3], Yusuph and Onumayi [4] and Onumayi et al. [5].

Block methods for solving ODEs have initially been proposed by Milne [6]. The Milne's idea of proceeding in blocks was developed by Rosser [7] for Runge-Kutta method. Also block Backward Differentiation Formulas (BDF) methods are discussed and developed by many researchers [8-16]. The method of collocation and interpolation of the power series approximate solution to generate continuous LMM has been adopted by many researchers among them are (Houwen et al. [17], Fatunla [18], Jiaxiang [19]).

In this paper we are suggested a construction of six step HBDF method, it is self-starting and can be applied for the numerical solution of IVPs (Cauchy problem) for second-order ODEs.

2 Materials and Methods

We seek an approximation of the form

$$Y(x) = \sum_{j=0}^{r+s-1} \ell_j x^j \tag{2}$$

Where ℓ_j are unknown coefficients to be determined and $k < r$ and $s > 0$ are the number of interpolation and collocation points respectively. We then construct our continuous approximation by imposing the following conditions:

$$Y(x) = y_{n+j}, \ j = 0,1,2,...,.k-1 \tag{3}$$

$$Y''(x_{n+k}) = f_{n+k} \tag{4}$$

We note that $y_{n+\mu}$ is the numerical approximation to the analytical solution $y(x_{n+\mu}), f_{n+\mu} = f(x_{n+\mu}, y_{n+\mu}, y'_{n+\mu})$.

Equations (3) and (4) lead to a system of (*k*+1) equations which is solved by Cramer's rule to obtain ℓ_j . Our continuous approximation is constructed by substituting the values ℓ_j into equation (2). After some manipulation, the continuous method is expressed as

$$Y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_\mu(x) y_{n+\mu} + h^2 \beta_k(x) f_{n+k} \tag{5}$$

Where $\alpha_j(x), \beta_k(x)$ and $\alpha_\mu(x)$ are continuous coefficients. We note that since the general second order ordinary differential equation involves the first derivative, the first derivative formula

$$Y'(x) = \frac{1}{h}\left(\sum_{j=0}^{k-1} \alpha'_j(x) y_{n+j} + \alpha'_\mu(x) y_{n+\mu} + h^2 \beta'_k(x) f_{n+k} \right) \tag{6}$$

$$Y'(x) = \delta(x) \tag{7}$$

$$Y'(a) = \delta_0 \tag{8}$$

2.1 Specification of Methods

2.1.1 Six step methods with one- off -step point at interpolation

To derive this methods, we use Eq.(5) to obtained a continuous 5-step HBDF method with the following specification : r=7,s=1,k=6. We also express $\alpha_j(x), \alpha_\mu(x)$ and $\beta_k(x)$ as a functions of t, where $t = \dfrac{x - x_n}{h}$ to obtain the continuous form as follows:

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + \alpha_5 y_{n+5} + \alpha_{\frac{11}{2}} y_{n+\frac{11}{2}} + h^2 \beta_6 f_{n+6} \tag{9}$$

Where

$$\alpha_0(t)=1-\frac{2223121}{850080}t+\frac{3014973}{1133440}t^2-\frac{3157433}{2266880}t^3+\frac{186435}{453376}t^4-\frac{156729}{2266880}t^5+\frac{13999}{2266880}t^6-\frac{773}{3400320}t^7$$

$$\alpha_1(t)=\frac{27775}{3864}t-\frac{179425}{15456}t^2+\frac{2088853}{278208}t^3-\frac{231881}{92736}t^4+\frac{126277}{278208}t^5-\frac{3961}{92736}t^6+\frac{227}{139104}t^7$$

$$\alpha_2(t)=-\frac{4125}{368}t-\frac{244245}{10304}t^2-\frac{1109779}{61824}t^3+\frac{409541}{61824}t^4-\frac{26609}{20608}t^5+\frac{7909}{61824}t^6-\frac{157}{30912}t^7$$

$$\alpha_3(t)=\frac{25619}{1932}t-\frac{389547}{12880}t^2+\frac{648047}{25760}t^3-\frac{51789}{5152}t^4+\frac{10755}{5152}t^5-\frac{5601}{25760}t^6+\frac{347}{38640}t^7$$

$$\alpha_4(t)=-\frac{57805}{5152}t-\frac{546379}{20608}t^2-\frac{8589991}{370944}t^3+\frac{403209}{41216}t^4-\frac{794791}{370944}t^5+\frac{9641}{41216}t^6-\frac{1865}{185472}t^7$$

$$\alpha_5(t)=\frac{2211}{280}t-\frac{21309}{1120}t^2+\frac{114827}{6720}t^3-\frac{10049}{1344}t^4+\frac{3817}{2240}t^5-\frac{1301}{6720}t^6+\frac{29}{3360}t^7$$

$$\alpha_{\frac{11}{2}}(t)=-\frac{17516}{5313}t-\frac{30368}{3795}t^2-\frac{247696}{34155}t^3+\frac{7312}{2277}t^4-\frac{5072}{6831}t^5+\frac{976}{11385}t^6-\frac{928}{239085}t^7$$

$$\beta_6(t)=\frac{1}{10304}\left(1320t-3254t^2+3023t^3-1385t^4+335t^5-41t^6+2t^7\right)$$

Evaluating (9) at $x=x_{n+6}$ yields Hybrid Six step implicit method

$$y_{n+6}=\frac{257}{28336}y_n-\frac{51}{644}y_{n+1}+\frac{405}{1288}y_{n+2}-\frac{247}{322}y_{n+3}+\frac{3555}{2576}y_{n+4}-\frac{81}{28}y_{n+5}+\frac{768}{253}y_{n+\frac{11}{2}}+\frac{45}{644}h^2f_{n+6} \tag{10}$$

Taking the second derivative of equation of equation (9), thereafter, evaluating the resulting continuous polynomial solution at $x=x_{n+2},x=x_{n+3},x=x_{n+4},x=x_{n+5},x=x_{n+\frac{11}{2}}$ we generate five additional methods

$$\begin{aligned}y_{n+2}=&-\frac{16829}{579590}y_n+\frac{81682}{142263}y_{n+1}+\frac{7724}{26345}y_{n+3}+\frac{30685}{94842}y_{n+4}\\&-\frac{7498}{26345}y_{n+5}+\frac{960512}{7824465}y_{n+\frac{11}{2}}-\frac{2576}{5269}h^2f_{n+2}-\frac{26}{5269}h^2f_{n+6}\end{aligned} \tag{11}$$

$$\begin{aligned}y_{n+3}=&\frac{353}{85976}y_n-\frac{2915}{52758}y_{n+1}+\frac{6455}{11724}y_{n+2}+\frac{39035}{70344}y_{n+4}-\frac{391}{5862}y_{n+5}\\&-\frac{3584}{290169}y_{n+\frac{11}{2}}-\frac{3220}{8793}h^2f_{n+3}+\frac{5}{17586}h^2f_{n+6}\end{aligned} \tag{12}$$

$$y_{n+4} = -\frac{11601}{15281585} y_n + \frac{8468}{833541} y_{n+1} - \frac{20754}{277847} y_{n+2} + \frac{774072}{1389235} y_{n+3} + \frac{908316}{1389235} y_{n+5} - \frac{6680576}{45844755} y_{n+\frac{11}{2}} - \frac{92736}{277847} h^2 f_{n+4} - \frac{540}{277847} h^2 f_{n+6} \tag{13}$$

$$y_{n+5} = \frac{2663}{3217148} y_n - \frac{16075}{1974159} y_{n+1} + \frac{5675}{146234} y_{n+2} - \frac{9514}{73117} y_{n+3} + \frac{1266205}{2632212} y_{n+4} + \frac{13411328}{21715749} y_{n+\frac{11}{2}} - \frac{560}{3179} h^2 f_{n+5} - \frac{5}{4301} h^2 f_{n+6} \tag{14}$$

$$y_{n+\frac{11}{2}} = -\frac{5200659}{563806208} y_n + \frac{11352935}{140951552} y_{n+1} - \frac{90099405}{281903104} y_{n+2} + \frac{54767691}{70475776} y_{n+3} - \frac{773233395}{563806208} y_{n+4} + \frac{259721451}{140951552} y_{n+5} + \frac{34155}{157312} h^2 f_{n+\frac{11}{2}} - \frac{7248285}{140951552} h^2 f_{n+6} \tag{15}$$

Since our method is design to simultaneously provide the values of $y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+\frac{11}{2}}, y_{n+6}$ at a block point $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+\frac{11}{2}}, x_{n+6}$, the six equations (10)- (15) are not sufficient to provide the solution for seven unknown. $y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+\frac{11}{2}}, y_{n+6}$. Thus, we obtain an additional method from (8), given by

$$850080 h\delta_0 + 2223121 y_0 - 6110500 y_1 + 9528750 y_2 - 11272360 y_3 + 9537825 y_4 - 6712596 y_5 + 2805760 y_{\frac{11}{2}} = 108900 h^2 f_6 \tag{16}$$

The derivatives are obtained from (7) by imposing that $\delta(x_{n+\mu}) = \delta_{n+\mu}, \mu = \{j, v\}, j = 0,...6,$ thus, we have

$$h\delta_{n+1} = -\frac{39411}{283360} y_n - \frac{17177}{11592} y_{n+1} + \frac{8037}{2576} y_{n+2} - \frac{8919}{3220} y_{n+3} + \frac{10747}{5152} y_{n+4} - \frac{387}{280} y_{n+5} + \frac{45184}{79695} y_{n+\frac{11}{2}} - \frac{27}{1288} h^2 f_{n+6}$$

$$h\delta_{n+2} = \frac{1803}{80960}y_n - \frac{1061}{3312}y_{n+1} - \frac{12865}{15456}y_{n+2} + \frac{327}{184}y_{n+3} - \frac{4441}{4416}y_{n+4} + \frac{143}{240}y_{n+5} - \frac{3776}{15939}y_{n+\frac{11}{2}} + \frac{3}{368}h^2 f_{n+6}$$

$$h\delta_{n+3} = -\frac{2833}{340032}y_n + \frac{695}{7728}y_{n+1} - \frac{2895}{5152}y_{n+2} - \frac{6493}{19320}y_{n+3} + \frac{11695}{10304}y_{n+4} - \frac{57}{112}y_{n+5} + \frac{5056}{26565}y_{n+\frac{11}{2}} - \frac{15}{2576}h^2 f_{n+6}$$

$$h\delta_{n+4} = \frac{309}{56672}y_n - \frac{607}{11592}y_{n+1} + \frac{633}{2576}y_{n+2} - \frac{2823}{3220}y_{n+3} + \frac{845}{15456}y_{n+4} + \frac{51}{56}y_{n+5} - \frac{22912}{79695}y_{n+\frac{11}{2}} + \frac{9}{1288}h^2 f_{n+6}$$

$$h\delta_{n+5} = -\frac{117}{25760}y_n + \frac{475}{11592}y_{n+1} - \frac{1325}{7728}y_{n+2} + \frac{297}{644}y_{n+3} - \frac{16435}{15456}y_{n+4} - \frac{197}{840}y_{n+5} + \frac{1408}{1449}y_{n+\frac{11}{2}} - \frac{15}{1288}h^2 f_{n+6}$$

$$h\delta_{n+6} = \frac{42859}{1700160}y_n - \frac{1693}{7728}y_{n+1} + \frac{4449}{5152}y_{n+2} - \frac{40189}{19320}y_{n+3} + \frac{37519}{10304}y_{n+4} - \frac{3849}{560}y_{n+5} + \frac{123328}{26565}y_{n+\frac{11}{2}} + \frac{801}{2576}h^2 f_{n+6}$$

$$h\delta_{n+\frac{11}{2}} = \frac{16209}{2072576}y_n - \frac{29315}{423936}y_{n+1} + \frac{184635}{659456}y_{n+2} - \frac{83259}{117760}y_{n+3} + \frac{258995}{188416}y_{n+4} - \frac{8217}{2048}y_{n+5} + \frac{498251}{159390}y_{n+\frac{11}{2}} + \frac{1485}{47104}h^2 f_{n+6}$$

2.2 Error Analysis and Zero Stability

Following Fatunla [18] and Lambert [20] we define the local truncation error associated with the conventional form of (5) to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k}\{\alpha_j y(x+jh)\} + \alpha_v y(x+vh) + h^2\beta_v y''(x+jh) \tag{17}$$

Assuming that y(x) is sufficiently differentiable, we can expand the terms in (17) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy' + \ldots, + C_q h^q y^q(x) + \ldots, \tag{18}$$

where the constant coefficients $C_q, \quad q = 0,1,\ldots$ are given as follows: $C_q, \quad q = 0,1,\ldots$

$$C_0 = \sum_{j=0}^{k} \alpha_j,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j,$$

$$.$$

$$.$$

$$.C_q = \left[\frac{1}{q!} \sum_{j=1}^{k} j^q \alpha_j - q(q-1) \sum_{j=1}^{k} j^{q-2} \beta_j \right].$$

According to Henrici [21], method (5) has order p if

$$C_0 = C_1 = ... = C_P = C_{P+1} = 0, \quad C_{P+2} \neq 0$$

Therefore, C_{p+2} is the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ the principal local truncation error at the point x_n. It is establish from our calculations that the HBDF have higher order and relatively small error constants as displayed in the Table 1.

Table 1. Order and error constants for the HBDF methods

Step number	Method	Order	Error constant
6	(9)	6	$-\frac{801}{144256}$
	(10)	6	$\frac{3446421}{56}$
	(11)	6	$-\frac{87945}{56}$
	(12)	6	$-\frac{141471}{56}$
	(13)	6	$-\frac{1378311}{56}$
	(14)	6	$-\frac{179315}{6924288}$
	(15)	6	$-\frac{31273}{288512}$

In order to analyze the methods for zero stability, we normalize the HBDF schemes and write them as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det\left(RA^{(0)} - A^{(1)}\right) = R^k (R-1) \tag{19}$$

Where $A^{(0)}$ is the identity matrix of dimension k+1, $A^{(1)}$ is the matrix of dimension k+1 Case k=6. It is easily shown that (9)-(16) are normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det\left(RA^{(0)} - A^{(1)}\right) = R^6 (R-1)$$

Where $A^{(0)}$ an identity matrix of is dimension seven and $A^{(1)}$ is a matrix of dimension seven given by

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Following Fatunla [18] the block method by combining k+1 HBDF is zero-stable, since from (19), $\rho(R) = 0$ satisfy $|R_j| \leq 1\ j = 1\ldots,k$ and for those roots with $|R_j|$=1, the multiplicity does not exceed 2. The block method by combining k+1 HBDF is consistent since HBDF have order $P > 1$. According to Henrici [21], we can safely ascertain the convergence of HBDF method.

3 Results

We report here a numerical example taken from the literature.

Problem 1

$$y'' - y' = 0,\ y(0) = 0, y'(0) = -1, h = 0.1$$

Exact Solution $y(x) = 1 - e^x$

Source: Mohammed [8]

Problems 2

$$y'' + y = 0,\ y(0) = 1, y'(0) = 1, h = 0.1$$

Exact Solution $y(x) = \cos(x) + \sin(x)$

Source: Awari [22]

Problems 3

$$y'' = x(y')^2,\ y(0) = 1, y'(0) = \frac{1}{2}, h = \frac{1}{30}$$

Exact Solution $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Source: Badmus and Yahaya [23]

4 Discussion

The HBDF methods are implemented as simultaneous numerical integration for IVPs without requiring starting values and predictors (Tables 2, 3 and 4). We proceed by explicitly obtaining initial conditions at x_{n+k} , n=0,k,….,N-k using the computed values $Y(x_{n\_k}) = y_{n+k}$ and $\delta(x_{n\_k}) = \delta_{n+k}$ over sub-intervals $[x_0, x_k],\ldots,[x_{N-K}, x_N]$ which do not overlap. We give examples to illustrate the efficiency of the methods.

We report here a numerical example taken from the literature.

Table 2. Showing exact solutions and the computed results from the proposed methods for problem 1

x	Exact solution	Proposed method	Error in proposed method	Error in mohammed [8]
0	0	0	1.4800000E-08	2.1980000E-05
0.1	-0.105170918	-0.1051709032	3.8100000E-08	6.0704000E-06
0.2	-0.221402758	-0.2214027199	6.2400000E-08	1.0051000E-05
0.3	-0.349858808	-0.3498587456	8.6200003E-08	1.4025300E-05
0.4	-0.491824698	-0.4918246118	1.1030000E-07	1.7993400E-05
0.5	-0.648721271	-0.6487211607	1.3360000E-07	2.1616200E-05
0.6	-0.822118800	-0.8221186664	1.5400000E-07	2.7993000E-05
0.7	-1.013752707	-1.013752553	1.8200000E-07	3.4561000E-05
0.8	-1.225540928	-1.225540746	2.1000000E-07	4.1114000E-05
0.9	-1.459603111	-1.459602901	2.3800000E-07	4.7656000E-05
1.0	-1.718281828	-1.718281590	1.4800000E-08	2.1980000E-05

Table 3. Showing exact solutions and the computed results from the proposed methods for problem 2

x	Exact solution	Proposed method	Error in proposed method	Error in awari [22]
0	1	1	0.0000000E+00	0.0000E-00
0.1	1.094837582	1.094837655	7.300000E-08	1.1570E-07
0.2	1.178735909	1.178736102	1.9300000E-07	3.0990E-07
0.3	1.250856696	1.250857010	3.1400000E-07	5.0550E-07
0.4	1.310479336	1.310479768	4.3200000E-07	6.9570E-07
0.5	1.357008100	1.357008646	5.4599999E-07	8.7890E-07
0.6	1.389978088	1.389978742	6.5400002E-07	1.0540E-06
0.7	1.409059874	1.409060598	7.2400000E-07	1.0080E-06
0.8	1.414062800	1.414063636	8.3600000E-07	9.2260E-07
0.9	1.404936878	1.404937018	1.4000000E-07	8.2610E-07
1.0	1.38177329	1.381774327	1.0370000E-07	7.2160E-07

Table 4. Showing exact solutions and the computed results from the proposed methods for problem 3

X	Exact value	Approx value	Present error	Yahaya and badmus [23]
0.1	1.050041729	1.050041724	5.00E-10	5.891E-06
0.2	1	1.100318692	1.67E-06	8.2399E-05
0.3	1.151140436	1.151028384	1.12E-05	3.46421E-04
0.4	1.202732554	1.202585545	1.47E-05	7.52101E-04
0.5	1.255412817	1.255265756	1.47E-05	1.380283E-03

5 Conclusion

In this paper we developed a uniform order 1-block 6 –point integrators of orders (6,6,6,6,6,6) and the resultant numerical integrators posses the following desirable properties.

(I) Zero-stability i.e stability at the origin
(II) Facility to generate the solution at six point simultaneously
(III) It is a convergence schemes

Hence, an improvement over other cited works.

Competing Interests

Authors have declared that no competing interests exist.

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