Fourth-order Four-stage Almost Runge-Kutta Methods for Initial Value Problems

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ABSTRACT
The process leading to the construction of Almost Runge-Kutta (ARK) methods is analysed, from whence two new methods of orders four with four stages are constructed after a thorough and careful assignment of values to the free parameters involved. Standard analysis established their convergence and thus effectiveness. Numerical experiments confirmed not just their effectiveness but also their efficiency as they proved to perform better than some existing methods.

Keywords: *convergence; stability; order; consistency*

INTRODUCTION

The explicit Almost Runge-Kutta (ARK) method for solution of the initial value problem

$$y' - f(x, y), \quad y(x_0) - y_0 \tag{1}$$

has the general form

Explicit Almost Runge-Kutta methods have the general form:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_s \\ \cdots \\ y_1^{[n]} \\ y_2^{[n]} \\ y_3^{[n]} \end{bmatrix} = \left[\begin{array}{cccccc|ccc} 0 & 0 & 0 & \cdots & 0 & 0 & & & \\ a_{21} & 0 & 0 & \cdots & 0 & 0 & & & \\ a_{31} & a_{32} & 0 & \cdots & 0 & 0 & e & c-Ae & \frac{c^2}{2}-Ac \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ a_{s-1,1} & a_{s-1,2} & a_{s-1,3} & \cdots & 0 & 0 & & & \\ b_1 & b_2 & b_3 & \cdots & b_{s-1} & 0 & & & \\ \hline b_1 & b_2 & b_3 & \cdots & b_{s-1} & 0 & 1 & b_0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{s-1} & \beta_s & 0 & \beta_0 & 0 \end{array}\right] \begin{bmatrix} hF_1 \\ hF_2 \\ \vdots \\ hF_s \\ \cdots \\ y_1^{[n-1]} \\ y_2^{[n-1]} \\ y_3^{[n-1]} \end{bmatrix} \tag{2}$$

where A is an $s \times s$ strictly lower triangular matrix, wherein $e_s^T = b^T$ with the implication that the last row of A is no different from the vector b. In similarity to Runge-Kutta (RK) methods, the vector b represents the weights and is of length s; c is equally a vector of length s representing the points at which the function f is evaluated; e and e_s are vectors of length s whose components are made up of entirely 1s and entirely zeros except the sth component which is 1, respectively (Abraham, 2010).

Since its introduction by [2], ARK methods have enjoyed a lot of contributions from various researchers, among whom are [4], [5], [6] and [7].

MATERIALS AND METHODS

According to Butcher [3], a fourth-order four-stage ARK method takes the general form

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & c_1 & \frac{1}{2}c_1^2 \\ a_{21} & 0 & 0 & 0 & 1 & c_2 - a_{21} & \frac{1}{2}c_2^2 - a_{21}c_1 \\ a_{31} & a_{32} & 0 & 0 & 1 & c_3 - a_{31} - a_{32} & \frac{1}{2}c_3^2 - a_{31}c_1 - a_{32}c_2 \\ b_1 & b_2 & b_3 & 0 & 1 & b_0 & 0 \\ \hline b_1 & b_2 & b_3 & 0 & 1 & b_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & \beta_0 & 0 \end{array}\right] \quad (3)$$

wherein the constituents of the first output approximations are given in (4) – (9)

$$b_0 + b^T c = 1 \quad (4)$$

$$b^T c = \frac{1}{2} \quad (5)$$

$$b^T c^2 = \frac{1}{3} \quad (6)$$

$$b^T c^3 = \frac{1}{4} \quad (7)$$

$$b^T A c^2 = \frac{1}{12} \quad (8)$$

$$b^T \left(\frac{1}{2}c^2 - Ac\right) = 0 \quad (9)$$

From (8) and (11),

$$b^T A c = \frac{1}{6} \quad (10)$$

It follows that,

$$\beta^T e + \beta_0 = 0 \quad (11)$$

$$\beta^T c = 1 \quad (12)$$

The Runge-Kutta stability conditions are therefore,

$$\beta^T (I + \beta_4 A) = \beta_4 e_4^T \quad (13)$$

$$c_1 = -2\frac{exp_4(-\beta_4)}{\beta_4 exp_3(-\beta_4)} \quad (14)$$

$$\left(1 + \frac{1}{2}\beta_4 c_1\right) b^T A^2 c = \frac{1}{4!} \quad (15)$$

Further computations result in

$$c_1 = -\frac{2\left(1 - \beta_4 + \frac{1}{2}\beta_4^2 - \frac{1}{6}\beta_4^3 - \frac{1}{24}\beta_4^4\right)}{\beta_4\left(1 - \beta_4 + \frac{1}{2}\beta_4^2 - \frac{1}{6}\beta_4^3\right)} \quad (16)$$

$$\left.\begin{array}{l} b_0 + b_1 + b_2 + b_3 = 1 \\ b_1 c_1 + b_2 c_2 + b_3 c_3 = \frac{1}{2} \\ b_1 c_1^2 + b_2 c_2^2 + b_3 c_3^2 = \frac{1}{3} \\ b_1 c_1^3 + b_2 c_2^3 + b_3 c_3^3 = \frac{1}{4} \end{array}\right\} \quad (17)$$

$$b_1 = \frac{6c_2c_3 - 4c_2 - 4c_3 + 3}{12c_1(c_1 - c_3)(c_1 - c_2)} \quad (18)$$

$$b_2 = \frac{6c_1c_3 - 4c_1 - 4c_3 - 3}{12c_2(c_1 - c_2)(c_2 - c_3)} \tag{19}$$

$$b_3 = \frac{6c_1c_2 - 4c_1 - 4c_2 + 3}{12c_3(c_2 - c_3)(c_1 - c_3)} \tag{20}$$

$$b_0 = \frac{12c_1c_2c_3 - 6c_1c_2 - 6c_2c_3 + 4c_1 + 4c_2 + 4c_3 - 3}{12c_1c_2c_3} \tag{21}$$

$$a_{21} = \frac{1}{12b_3a_{32}c_1(2 + \beta_1c_1)} \tag{22}$$

$$a_{31} = \frac{\frac{1}{6} - b_3a_{32}c_2 - b_2a_{21}c_1}{b_3c_1} \tag{23}$$

$$a_{32} = \frac{1 - 2c_1}{12b_3c_2(c_2 - c_1)} \tag{24}$$

$$\beta_1 = -a_{31}b_3\beta_4^3 + a_{21}b_2\beta_4^3 - a_{21}a_{32}b_3\beta_4^4 - b_1\beta_4^2 \tag{25}$$

$$\beta_2 = a_{32}b_3\beta_4^2 - b_2\beta_4^2 \tag{26}$$

$$\beta_3 = -b_3\beta_4^2 \tag{27}$$

$$\beta_4 = \beta_4 \tag{28}$$

$$\beta_0 = -\beta_1 - \beta_2 - \beta_3 - \beta_4 \tag{29}$$

Upon computation of the elements of matrix U the following two schemes named ARK4a and ARK4b are obtained

ARK4a with $c^T = \left[\frac{11}{24}, \frac{1}{2}, \frac{1}{3}, 1\right]$, $\beta_4 = 3$:

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & \frac{11}{24} & \frac{121}{1152} \\ \frac{16}{99} & 0 & 0 & 0 & 1 & \frac{67}{198} & \frac{11}{216} \\ -\frac{64}{143} & \frac{2}{39} & 0 & 0 & 1 & \frac{313}{429} & \frac{55}{234} \\ -\frac{256}{11} & 18 & \frac{13}{2} & 0 & 1 & -\frac{5}{22} & 0 \\ \hline -\frac{256}{11} & 18 & \frac{13}{2} & 0 & 1 & -\frac{5}{22} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{2256}{11} & -153 & -\frac{117}{2} & 3 & 0 & \frac{75}{22} & 0 \end{array}\right] \tag{30}$$

ARK4b with c^T $\left[1, \frac{1}{3}, \frac{3}{4}, 1\right]$, $\beta_1 = 3$:

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & \frac{11}{24} & \frac{121}{1152} \\ -\frac{32}{99} & 0 & 0 & 0 & 1 & \frac{65}{99} & \frac{11}{54} \\ \frac{364}{297} & -\frac{35}{144} & 0 & 0 & 1 & -\frac{1105}{4752} & -\frac{517}{2592} \\ -\frac{64}{77} & \frac{11}{10} & \frac{24}{35} & 0 & 1 & \frac{1}{22} & 0 \\ \hline -\frac{64}{77} & \frac{11}{10} & \frac{24}{35} & 0 & 1 & \frac{1}{22} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1248}{77} & -\frac{72}{5} & -\frac{216}{35} & 3 & 0 & \frac{15}{11} & 0 \end{array}\right] \quad (31)$$

Convergence Analysis

A fundamental criteria for convergence of a numerical method is that the method be consistent and stable. A method of order at least 1 is necessarily consistent. The two methods ARK4a and ARK4b are both of order $p = 4 > 1$; thus they are consistent. Stability of multivalued method s is guaranteed if all positive powers of the V matrix have bounded members. To satisfy this, it is sufficient that the absolute value of every eigenvalue of V, save the Principal Eigenvalue (which equals 1), be less than 1.

For ARK methods, V is a 3×3 matrix whose eigenvalues are $\lambda = 1,0,0$. Hence the powers of V are bounded. This can be further illustrated using the Cay ley–Hamilton theorem which states that every square matrix satisfies its characteristic equation . In the case of ARK methods, it implies $V^3 - V^2 = 0$, or $V^3 = V^2$, or $V^n = V^2$ for every n greater than 2. Hence V^n is bounded.

Numerical Experiments

Problems 1 and 2 are sample problems that are solved by the methods ARK4a and ARK4b. The results are obtained and compared with those of existing methods of similar order to ascertain their effectiveness. The results are further presented in Tables 1 and 2.

Problem 1

$y' = x + y, \quad y(0) = 1$

$h = 0.1, \quad x \in [0, 1]$

$y(x) = 2e^x - x - 1$

Problem 2

$y' = x + 2y, \quad y(0) = 1$

$h = 0.1, \quad x \in [0, 1]$

$y(x) = -\frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{2x}$

TABLE 1 RESULTS OF PROBLEM 1

x	y_{exact}	$y_{([7])}$	$Error_{([7])}$	$y_{(ARK4a)}$	$Error_{(ARK4a)}$	$y_{(ARK4b)}$	$Error_{(ARK4b)}$
0.0	1.000000000	1.000000000	0.000000000	1.000000000	0.000000000	1.000000000	0.000000000
0.1	1.110341836	1.110341764	0.000000072	1.110341783	0.000000053	1.110341781	0.000000055
0.2	1.242805516	1.242805249	0.000000266	1.242805272	0.000000244	1.242805269	0.000000247
0.3	1.399717615	1.399717113	0.000000502	1.399717139	0.000000476	1.399717135	0.000000480
0.4	1.583649395	1.583648612	0.000000783	1.583648643	0.000000752	1.583648636	0.000000759
0.5	1.797442541	1.797441422	0.000001118	1.797441461	0.000001080	1.797441449	0.000001092
0.6	2.044237600	2.044236085	0.000001516	2.0044236129	0.000001472	1.044236115	0.000001486
0.7	2.327505414	2.327503431	0.000001984	2.327503481	0.000001934	2.327503464	0.000001951
0.8	2.651081857	2.651079323	0.000002534	2.651079382	0.000002475	2.651079360	0.000002497
0.9	3.019206222	3.019203044	0.000003177	3.019203110	0.000003112	2.019203085	0.000003137
1.0	3.436563656	3.436559728	0.000003928	3.436569804	0.000003853	3.436559772	0.000003885

TABLE 2 RESULTS OF PROBLEM 2

x	y_{exact}	$y_{([7])}$	$Error_{([7])}$	$y_{(ARK4a)}$	$Error_{(ARK4a))}$	$y_{(ARK4b)}$	$Error_{(ARK4b)}$
0.0	1.000000000	1.000000000	0.000000000	1.000000000	0.000000000	1.000000000	0.000000000
0.1	1.226753448	1.226751953	0.000001495	1.226752264	0.000001184	1.226752263	0.000001185
0.2	1.514780872	1.514774835	0.000006037	1.514775217	0.000005655	1.514775216	0.000005656
0.3	1.877648500	1.877635984	0.000012516	1.877636452	0.000012048	1.877636451	0.000012049
0.4	2.331926160	2.331904590	0.000021570	2.331905164	0.000020996	2.331905163	0.000020997
0.5	2.897852285	2.897818267	0.000034018	2.897818980	0.000033305	2.897818979	0.000033306
0.6	3.600146154	3.600095232	0.000050922	3.600096109	0.000050045	3.600096108	0.000050046
0.7	4.468999959	4.468926316	0.000073643	4.468927391	0.000072568	4.468927390	0.000072569
0.8	5.541290530	5.541186602	0.000103928	5.541187912	0.000102618	5.541187911	0.000102619
0.9	6.862059330	6.861915315	0.000144015	6.861916953	0.000142377	6.861916952	0.000142378
1.0	8.486320124	8.486123367	0.000196757	8.48612537	0.000194754	8.48612536	0.000194764

Discussion of Results

In Tables 1 and 2, the results of the two fourth order methods, ARK4a (designated as $y_{(ARK4a)}$) and ARK4b (designated as $y_{(ARK4b)}$) exhibit less errors (designated as $Error_{(ARK4a)}$ and $Error_{(ARK4b)}$) than the fourth order method of [7] (designated as $y_{([7])}$) through the domain of integration.

Conclusion

This research has succeeded in deriving two fourth order four stage Almost Runge-Kutta methods for the solution of initial value problems of ordinary differential equations. The methods have proven to be not just effective but also efficient as both of them produce lesser errors when compared to some existing methods.

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