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ONE-STEP SECOND DERIVATIVE BLOCK INTRA-STEP METHOD FOR STIFF SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Presented here is a one-step second derivative intra-point block numerical method of uniform order eight for seeking the solution of stiff systems of ordinary differential equations directly. The techniques of interpolation and collocation are used to build the continuous scheme from which the main method and additional methods were obtained. The convergence analysis of the proposed method to validate their effectiveness and reliability is presented. Some numerical examples both linear and non-linear are solved to demonstrate the efficiency of the method.

Keywords and phrases: collocation, interpolation, intra-point, second derivative, single-step
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1. INTRODUCTION

One of the most powerful tools for modeling in science and engineering is ordinary differential equations (ODEs). That defines its importance in the said areas. As important as ODEs, its solution is more important, as its interpretation relies solely on the solution.In many physical problems modeled by systems of ordinary differential equations do exhibit a behavior known as stiffness [1]. Curtiss and Hirschefelder [2] evented the knowledge of stiffness occurring in differential equations. Shampine and Gear [3] expounded the characteristics of numerical methods used for solving problems with stiffness and discussed the different realistic goals when solving stiff problems which involves methods with strong stability properties for solving stiff systems.

The common methods used to solve ODEs are categorized as single-step (multistage) methods such as Runge-Kutta methods and

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multistep (one-stage) methods such as Adams-Bashforth-Moulton methods [4]. Morrison and Stroller [5], Butcher [6], Fang [7], Watts and Shampine [8] have fully studied implicit one-step methods. We will consider a class of block hybrid second derivative implicit Euler method for solving ODEs constructed through interpolation and collocation techniques, see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. It is worth noting that the second derivative methods to be of order up to $k+2$ were proposed by Enright [21]. This method usually obtained a block of new values simultaneously which makes its computation to be competitive, and can be used in solving stiff ODEs effectively.

The method preserves the Rung-kutta traditional advantage of being self-starting and is more accurate since it is implemented as a block method. We emphasis that block methods were first introduced by Milne [22] for the purpose of obtaining starting values for predictor-corrector algorithms, see [23]. However, Rosser [24] developed Milnes idea into algorithms for general use. We emphasis that the continuous representation of our method is developed for general use, not only as a means of obtaining starting values for predictor-corrector algorithms; it generates a main discrete scheme and two additional method which are combined and implemented as a block method and simultaneously generates approximations $\{y_{n+\frac{1}{4}}, y_{n+\frac{3}{4}}, y_{n+1}\}$ to exact solutions $\{y(x_{n+\frac{1}{4}}), y(x_{n+\frac{3}{4}}), y(x_{n+1})\}$. In this paper, we seek to construct and investigate one-step second derivative block intra-step method for stiff system of ordinary differential equation, which takes the form

$$y' = f(x, y), \quad y(x_0) = y_0, \quad \forall x_0 \leq x \leq x_N, \tag{1}$$

Where function $f(x, y)$ is continuous within the interval of integration and existence of a unique solution is assumed.

Section 2 is devoted to formulation of the proposed method. The analysis of the methods such as local truncation error, order and error constant, consistent, convergence and stability region of the proposed method are presented in Section 3. Numerical experiments are presented in Section 4. And conclusion is given Section 5.

2. DERIVATION OF THE METHOD

The proposed one-step second derivative block intra-step method (OSDBM) for approximating the analytic solution of (1) is derived by seeking a polynomial of degree 8 as an approximate solution of the form

$$Y(x) = \sum_{j=0}^{8} a_j x^j \tag{2}$$

where a_js are unknown coefficients to be determined, and we emphasis that (2) satisfies the system of nine equations below

$$Y(x_n) = y_n, \tag{3}$$

$$Y'(x_{n+j}) = \sum_{j=0}^{8} j a_j x_{n+j}^{j-1} = f_{n+j}, \quad j = 0, \frac{1}{4}, \frac{3}{4}, 1 \tag{4}$$

$$Y''(x_{n+j}) = \sum_{j=0}^{8} j(j-1) a_j x_{n+j}^{j-2} = g_{n+j}, \quad j = 0, \frac{1}{4}, \frac{3}{4}, 1, \tag{5}$$

n is the grid index and the second derivative in Equation (5) coincides with the second derivative of the analytical solution at mesh points x_{n+j}, $j = 0, \frac{1}{4}, \frac{3}{4}, 1$. Equations (3)-(5) lead to a system of nine equations whose matrix form is given by

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 0 & 1 & 2\,x_n & 3\,x_n^2 & 4\,x_n^3 & 5\,x_n^4 & 6\,x_n^5 & 7\,x_n^6 & 8\,x_n^7 \\ 0 & 1 & 2x_{n+\frac{1}{4}} & 3x^2_{n+\frac{1}{4}} & 4x^3_{n+\frac{1}{4}} & 5x^4_{n+\frac{1}{4}} & 6x^5_{n+\frac{1}{4}} & 7x^6_{n+\frac{1}{4}} & 8x^7_{n+\frac{1}{4}} \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{1}{4}} & 4x^3_{n+\frac{1}{4}} & 5x^4_{n+\frac{1}{4}} & 6x^5_{n+\frac{1}{4}} & 7x^6_{n+\frac{1}{4}} & 8x^7_{n+\frac{1}{4}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} & 7x^7_{n+1} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12x^2_{n+\frac{1}{4}} & 20x^3_{n+\frac{1}{4}} & 30x^4_{n+\frac{1}{4}} & 42x^5_{n+\frac{1}{4}} & 56x^6_{n+\frac{1}{4}} \\ 0 & 0 & 2 & 6x_{n+\frac{3}{4}} & 12x^2_{n+\frac{3}{4}} & 20x^3_{n+\frac{3}{4}} & 30x^4_{n+\frac{3}{4}} & 42x^5_{n+3\frac{3}{4}} & 56x^6_{n+\frac{3}{4}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^5_{n+1} & 56x^6_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \\ g_n \\ g_{n+\frac{1}{4}} \\ g_{n+\frac{3}{4}} \\ g_{n+1} \end{bmatrix}$$

and whose solution by matrix inversion method generates the coefficients a_j which are substituted into equation (2) to yield the continuous form of the one-step second derivative block intra-step method as

$$\begin{aligned} Y(x) &= y_n + h(\beta_0(x) f_n + \beta_{\frac{1}{4}}(x) f_{\frac{1}{4}} + \beta_{\frac{3}{4}}(x) f_{\frac{3}{4}} + \beta_1(x) f_{n+1}) \\ &+ h^2(\gamma_0(x) g_n + \gamma_{\frac{1}{4}}(x) g_{\frac{1}{4}} + \gamma_{\frac{3}{4}}(x) g_{\frac{3}{4}} + \gamma_1(x) g_{n+1}) \end{aligned} \tag{6}$$

where $\beta_j(x)$ and $\gamma_j(x)$, $j = 0, \frac{1}{4}, \frac{3}{4}, 1$ are continuous coefficients and h is the step size. We emphasis that $y_{n+j} = Y(x_{n+j})$ is the numerical approximation to the analytical solution $y(x_{n+j})$, $y'_{n+j} = f_{n+j}$ is the approximation to $y'(x_{n+j})$, and $g_{n+j} = \frac{d}{dx} f(x, y(x))|_{x_{n+j}}$. Evaluating (6) at $x = x_n + jh$, $j = 0, \frac{1}{4}, \frac{3}{4}, 1$ gives the one-step second derivative hybrid block method (OSDBM) as follows

$$\begin{aligned}
y_{n+\frac{1}{4}} &= y_n + \frac{20135\,hf_n}{193536} + \frac{3413\,hf_{n+\frac{1}{4}}}{24192} + \frac{11\,hf_{n+\frac{3}{4}}}{24192} + \frac{857\,hf_{n+1}}{193536} \\
&+ \frac{233\,h^2g_n}{71680} - \frac{1601\,h^2g_{n+\frac{1}{4}}}{161280} - \frac{289\,h^2g_{n+\frac{3}{4}}}{161280} - \frac{23\,h^2g_{n+1}}{71680} \\
y_{n+\frac{3}{4}} &= y_n + \frac{1125\,hf_n}{7168} + \frac{303\,hf_{n+\frac{1}{4}}}{896} + \frac{177\,hf_{n+\frac{3}{4}}}{896} + \frac{411\,hf_{n+1}}{7168} \\
&+ \frac{489\,h^2g_n}{71680} + \frac{423\,h^2g_{n+\frac{1}{4}}}{17920} - \frac{633\,h^2g_{n+\frac{3}{4}}}{17920} - \frac{279\,h^2g_{n+1}}{71680} \qquad (7) \\
y_{n+1} &= y_n + \frac{61\,hf_n}{378} + \frac{64\,hf_{n+\frac{1}{4}}}{189} + \frac{64\,hf_{n+\frac{3}{4}}}{189} + \frac{61\,hf_{n+1}}{378} \\
&+ \frac{h^2g_n}{140} + \frac{8\,h^2g_{n+\frac{1}{4}}}{315} - \frac{8\,h^2g_{n+\frac{3}{4}}}{315} - \frac{h^2g_{n+1}}{140}.
\end{aligned}$$

3. ANALYSIS OF OSDBM

We first write (7) in a block form given by the matrix difference equation

$$\begin{aligned}
P^{(1)}Y_w &= P^{(0)}Y_{w-1} \qquad (8)\\
&+ h(Q^{(0)}F_{w-1} + Q^{(1)}F_w) + h^2(R^{(0)}G_{w-1} + R^{(1)}G_w)
\end{aligned}$$

where $Y_w = (y_{n+\frac{1}{4}}, y_{n+\frac{3}{4}}, y_{n+1})^\top$, $Y_{w-1} = (y_{n-\frac{3}{4}}, y_{n-\frac{1}{4}}, y_n)^\top$, $F_w = (f_{n+\frac{1}{4}}, f_{n+\frac{3}{4}}, f_{n+1})^\top$, $F_{w-1} = (f_{n-\frac{3}{4}}, f_{n-\frac{1}{4}}, f_n)^\top$, $G_w = (g_{n+\frac{1}{4}}, g_{n+\frac{3}{4}}, g_{n+1})^\top$, $G_{w-1} = (g_{n-\frac{3}{4}}, g_{n-\frac{1}{4}}, g_n)^\top$, and the matrices $P^{(1)}$, $P^{(1)}$, $Q^{(1)}$, $Q^{(0)}$, $R^{(1)}$ and $R^{(0)}$ are 3×3 matrices whose entries are given by the coefficient of Equation (7) defined as follows:

$$P^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q^{(0)} = \begin{bmatrix} 0 & 0 & \frac{20135}{193536} \\ 0 & 0 & \frac{1125}{7168} \\ 0 & 0 & \frac{61}{378} \end{bmatrix}, \quad Q^{(1)} = \begin{bmatrix} \frac{3413}{24192} & \frac{11}{24192} & \frac{857}{193536} \\ \frac{303}{896} & \frac{177}{896} & \frac{411}{7168} \\ \frac{64}{189} & \frac{64}{189} & \frac{61}{378} \end{bmatrix},$$

$$R^{(0)} = \begin{bmatrix} 0 & 0 & \frac{233}{71680} \\ 0 & 0 & \frac{489}{71680} \\ 0 & 0 & \frac{1}{140} \end{bmatrix}, \quad R^{(1)} = \begin{bmatrix} -\frac{1601}{161280} & -\frac{289}{161280} & -\frac{23}{71680} \\ \frac{423}{17920} & -\frac{633}{17920} & -\frac{279}{71680} \\ \frac{8}{315} & -\frac{8}{315} & -\frac{1}{140} \end{bmatrix}.$$

3.1. Zero stability

Zero-stability has to do with the stability of the difference system (8) in the limit as h tends to 0 [1]. So, as $h \to 0$, the method (7) tends to the difference system

$$P^{(1)}Y_w - P^{(0)}Y_{w-1} = 0 \tag{9}$$

with its first characteristic polynomial $\rho(\lambda)$ is given as

$$\rho(\lambda) = |\lambda A^{(1)} - A^{(0)}|, \rho(\lambda) = \lambda^2(\lambda - 1) = 0, \lambda = \{0, 0, 1\}. \tag{10}$$

The OSDBM is zero stable as solution (10) satisfies that the roots $|\lambda_j| \leq 1,\ j = 0, \frac{1}{4}, \frac{3}{4}, 1$ and for those roots with $|\lambda_j| = 1$, the multiplicity does not exceed 1 [25].

3.2. Order and error constants:

Extending the definition of Fatunla [25] and Lambert [9] we express the local truncation error associated with the conventional form of (7) to be the linear difference operator

$$\begin{aligned} L[y(x); h] &= y_n + h(\beta_0(x)y'_n + \beta_{\frac{1}{4}}(x)y'_{\frac{1}{4}} + \beta_{\frac{3}{4}}(x)y'_{\frac{3}{4}} + \beta_1(x)y'_{n+1}) \\ &+ h^2(\gamma_0(x)y''_n + \gamma_{\frac{1}{4}}(x)y''_{\frac{1}{4}} + \gamma_{\frac{3}{4}}(x)y''_{\frac{3}{4}} + \gamma_1(x)y''_{n+1}). \end{aligned} \tag{11}$$

Assuming that $y(x)$ is sufficiently differentiable, we expand each term in Equation (11) as a Taylor series about the point to obtain the expression

$$L[y(x); h] = C_0 y(x) + C_1 h y(x) + \ldots + C_q h^q y(x) + \ldots \tag{12}$$

where the constant C_q, $q = 0, 1, \ldots$ are given as follows

$$\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_j, \\ C_1 &= \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j, \\ C_2 &= \frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j - \sum_{j=1}^{k} j\beta_j - \sum_{j=0}^{k} j\gamma_j, \quad \ldots, \\ C_q &= \frac{1}{q!}\sum_{j=1}^{k} j^q\alpha_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\gamma_j, \\ q &= 3, 4, \ldots. \end{aligned}$$

According to Henrici [26], the proposed OSDBM has a uniform order $p = (8, 8, 8)^\top$ with relative small error constant

$$C_{p+1} = C_9 = \left(\frac{103}{416179814400}, \frac{9}{5138022400}, \frac{13}{6502809600}\right)^\top.$$

3.3. Consistency and convergence

The OSDBM is consistent since each of the discrete schemes has order According to Henrici [26], the OSDBM is convergent, since the necessary and sufficient condition for convergence is for the method to be zero-stable and consistent.

3.4. Stability analysis

Following Brugnano and Trigiant [27] and Hairer and Wanner [28] we apply the usual test equations

$$y = \lambda y, \quad y'' = \lambda^2 y$$

to yield $Y_w = \sigma(z)Y_w - 1$, $z = \lambda h$ where the matrix $\sigma(z)$ is given by

$$\sigma(t, z) = |(P^{(1)} - zQ^{(1)} - z^2R^{(1)} - Q^{(0)} - R^{(0)})t + P^{(0)}|. \qquad (13)$$

From Equation (13) we obtain the stability function $R(z) : \mathbf{C} \to \mathbf{C}$ which is a rational function with real coefficients given by

$$R(z) = \frac{n(z)}{d(z)} \qquad (14)$$

where

$$\begin{aligned} n(z) &= 4(1521z^4 + 24576z^3 + 282432z^2 + 1310720z + 3870720), \\ d(z) &= 27\,z^6 - 1026\,z^5 + 19143\,z^4 - 246240\,z^3 \\ &+ 1636032\,z^2 - 8279040\,z + 12873728. \end{aligned}$$

Equation (14) is used to plot the stability region which lies outside the bounded region as shown in fig. 1 below which shows that the method is A-stable.

4. NUMERICAL EXPERIMENTS

Our method is implemented as a block for initial value problems which requires no starting values and predictors. We obtain initial conditions at x_{n+1}, $n = 0, 1, \ldots, N-1$ using the computed values y_n over sub-intervals $[x_0, x_1], \ldots, [x_{N-1}, x_N]$. For instance, when $n = 0, (y_{\frac{1}{4}}, y_{\frac{3}{4}}, y_1)$ are obtained simultaneously over the sub-interval $[x_0, x_1]$, as y_0 is known from the IVP, for $n = 1, (y_{\frac{5}{4}}, y_{\frac{7}{4}}, y_2)$ are also obtained simultaneously over the sub-interval $[x_1, x_2]$, as

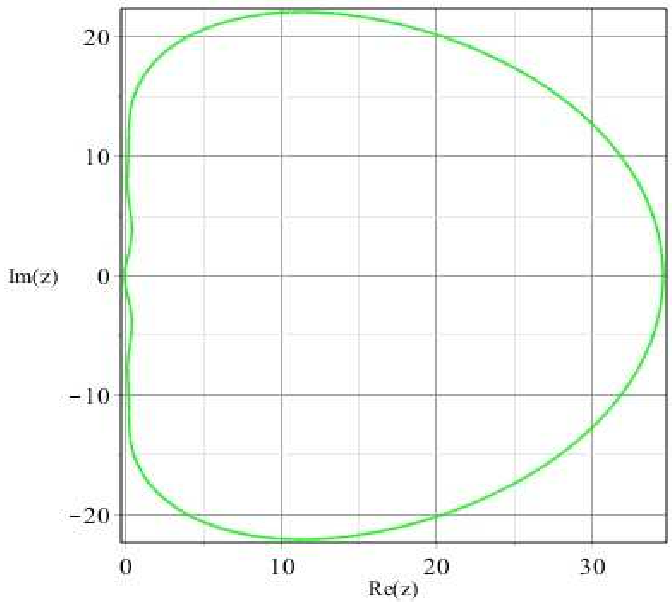

Fig. 1. Stability region.

y_1 is now known from the previous block, and so on. Therefore, the sub-interval $[x_n, x_{n-1}]$ does not over-lap and the solutions obtained in this manner are more accurate than those obtained in the conventional way.

For the purpose of comparative analysis of performance of the new method on the various numerical examples, we use the following notations:

(1) OSDBM: the proposed one-step second derivative block method of order $p = 8$ with two intra-points,
(2) NJ: one-step method of order $p = 6$ [30],
(3) AAK: two-step method of order $p = 8$ [31],
(4) AU: fitting numerical integrator of order $p = 9$ [32]
(5) MOS: two-step method of order $p = 4$ [33],
(6) AAO: three-step method of order $p = 8$ [1],
(7) HBSDBDF: three-step method of order $p = 7$ [29].

Problem 1. We consider the following linear system of IVP over the range $0 \leq x \leq 1$.

$$\begin{cases} y' = -y + 95z, & y(0) = 1, \\ z' = -y + 97z, & z(0) = 1. \end{cases}$$

Exact solution: $y(x) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x}$, $z(x) = \frac{48}{47}e^{-96x} - \frac{1}{47}e^{-2x}$.

This problem was solved in studies [31, 30, 32].The absolute error at $x = 1$ was obtained and compared as shown in Table 1. It is seen that the method performs better than existing methods.

Table 1. Absolute errors for Problem 1.

h	NJ $y(1)$ $z(1)$	AAK $y(1)$ $z(1)$	AU $y(1)$ $z(1)$	OSDBM $y(1)$ $z(1)$
0.06250	3×10^{-12}	1×10^{-16}	5×10^{-8}	6.5×10^{-17}
	3×10^{-12}	1×10^{-17}	7×10^{-10}	6.9×10^{-19}
0.03125	5×10^{-14}	5×10^{-19}	6×10^{-8}	2.5×10^{-19}
	5×10^{-14}	5×10^{-20}	1×10^{-10}	2.7×10^{-21}

Problem 2. We also considered the Van Der Pol Oscillatory Problem [33].

$$y'' - 2\xi\left(1-y^2\right)y' + y = 0, \quad y(0) = 0, \quad y'(0) = 0.5,$$
$$x \in [0,10], \quad \xi = 0.025.$$

The above second order differential equation is reduced to its corresponding first order system of equations:

$$\begin{cases} y' = z, & y(0) = 0, \\ z' = -y + 2\xi(1-y^2)z, & z(0) = 0.5. \end{cases}$$

The solution to this problem is validated with RK45 method in Maple Software and paper [33] given in the Table 2.

Table 2. Result for the Problem 2 with $h = 0.1$.

x	RK4	MOS	OSDBM
0.0	0	0	0
1.0	0.431051	0.431051	0.427460
2.0	0.476310	0.476309	0.468800
3.0	0.076077	0.076076	0.073843
4.0	-0.415460	-0.415460	-0.40320
5.0	-0.538570	-0.538570	-0.51831
6.0	-0.161350	-0.161340	-0.15320
7.0	0.386024	0.386025	0.36698
8.0	0.595231	0.595230	0.56047
9.0	0.254655	0.254653	0.23673
10.0	-0.341570	-0.341580	-0.31856

Problem 3. We also considered the nonlinear system of stiff differential equation in the range $0 \le x \le 10$ [1].

$$\begin{cases} y' = \mu y + z^2, & y(0) = -\frac{1}{\mu+2}, \\ z' = -z, & z(0) = 1. \end{cases}$$

Where $\mu = 10,000$, the exact solution is $y(x) = -\frac{e^{-2x}}{\mu+2}$, $z(x) = e^{-x}$.

From Table 3, the numerical results revealed that our method is superior in terms of accuracy when compared with the method $\frac{3}{8}$-type block method for stiff systems in [1].

Table 3. Comparative analysis of absolute error for Problem 3.

x	Error in AAO $h = 0.01$	OSDBM $h = 0.1$	Error in AAO $h = 0.01$	OSDBM $h = 0.1$
	y		z	
3	2.03×10^{-19}	3.79×10^{-22}	1.44×10^{-14}	2.998×10^{-18}
5	1.20×10^{-20}	1.60×10^{-21}	3.21×10^{-15}	6.74×10^{-19}
10	1.11×10^{-20}	7.12×10^{-20}	4.38×10^{-17}	9.08×10^{-21}

Problem 4. We consider the following nonlinear IVP over the range $0 \leq x \leq 10$ [29].

$$\begin{cases} y' = -1002y + 1000z^2, & y(0) = 0, \\ z' = y - z(1+z) & z(0) = 1. \end{cases}$$

Exact solution: $y(x) = e^{-2x}$, $z(x) = e^{-x}$.

Table 4. Comparative analysis of absolute error for Problem 4 at point $x = 10$.

h	HBSDBDF		OSDBM	
	y	z	y	z
2.5	2.1670×10^{-9}	1.35068×10^{-5}	5.4282×10^{-9}	1.7730×10^{-9}
1.25	2.3339×10^{-9}	2.8914×10^{-5}	$9.8036 \times 10 \times -11$	1.6315×10^{-10}
0.83333	2.3078×10^{-9}	2.9695×10^{-5}	$5.3933 \times 10 \times -12$	2.2141×10^{-13}
0.625	2.2987×10^{-9}	2.9986×10^{-5}	$4.4710 \times 10 \times -13$	2.2039×10^{-13}
0.5	2.2948×10^{-9}	3.0115×10^{-5}	$4.2640 \times 10 \times -14$	3.7130×10^{-15}

Problem 5. We also consider the following nearly sinusoidal problem [29].

$$\begin{cases} y' = -21y + z + 2\sin x, & y(0) = 2, \\ z' = 998y - 999z + 999\cos x - 999\sin x, & z(0) = 3. \end{cases}$$

Exact solution: $y(x) = 2e^{-x} + \sin x$, $z(x) = 2e^{-x} + \cos x$.

Table 5. Comparative analysis of absolute error for Problem 5.

h	HBSDBDF		OSDBM	
	Maximum error	Relative error	Maximum error	Relative error
0.4	8.9924×10^{-7}	3.6279×10^{-7}	9.819×10^{-13}	9.161×10^{-13}
0.2	5.9042×10^{-9}	2.6294×10^{-9}	3.542×10^{-15}	1.358×10^{-13}
0.1	4.5695×10^{-11}	1.8848×10^{-11}	1.743×10^{-17}	6.4596×10^{-16}

4. CONCLUDING REMARKS

In this paper, we proposed a new One-Step Second Derivative Block Intra-Step Method for Stiff Systems of Ordinary Differential Equations. The method is used together with additional methods in block form to simultaneously solve IVPs. The block method is found to be A-stable and implemented without the need for starting values or predictors and hence it is self-starting. We have demonstrated the efficiency of the method on five numerical examples both linear and non-linear stiff systems. From the results obtained, the accuracy of OSDBM is higher than those found in the literature. Tables 1-5 show the detail of the results. The results have shown that OSBDM is suitable for solving stiff problems and converges accurately even with large step size h.

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