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# A Class of Implicit Five Step Hybrid Method for Solving First Order Initial Value Problem

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Abstract

In this paper, a self-starting five step continuous block hybrid formulae (CBHF) with one Off-step point is developed using collocation and interpolation procedures. The CBHF is then used to produce multiple numerical integrators that are of uniform order and are assembled in to a single block matrix equation. These equations are simultaneously applied to provide the approximate solution for the ordinary differential equations. The order of accuracy of the block method was discussed and its accuracy over the methods in literature is established numerically. The result of the proposed hybrid method performed absolutely better, when compared with the exact solution and the result obtained by Mohammed (2011), which are all of step five (i.e. $k = 5$).

## 1. Introduction

(Lie and Norsett, 1989), (Onumanyi et al 1994), (Yahaya and Mohammed, 2010), (Yahaya, 2004) and (Mohammed, 2010) have all converted conventional linear

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multistep methods including hybrid ones into continuous forms through the idea of Multistep Collocation (MC). The Continuous Multistep (CM) method, associated with conventional linear multistep methods produces piece-wise polynomial solutions over $k-$steps for the first order differential system.
This research work aims at deriving a five-step block hybrid method for numerical integration of ordinary differential equations. It allows the block formulation and therefore is self-starting and for appropriate choice of $k$, overlap of solution model is eliminated.

## 2. The Method:

Let us consider the first order system of ODE's:

$$y' = f\left(x, y\right), \quad a \leq x \leq b \tag{1}$$

where $y$ satisfies some additional two-points or multi-point boundary conditions which can involve derivative values either the point $a, b$ or other points in between as well. The function $f$ is sufficiently smooth, $f : R^{m+1} \rightarrow R^{m}$. where $y$ is an m-dimensional vector and $x$ is a scalar variable.
A particular useful class of methods for (1) is the k-step linear multistep methods (LMMs) of the form:

$$\sum_{j=0}^{k} \phi_j y_{n+j} = h \sum_{j=0}^{k} \varphi_j f_{n+j} \tag{2}$$

The idea of the k-step MC, following (Onumanyi et al, 1994), is to find a polynomial $U$ of the form:

$$U = \sum_{j=0}^{t-1} \phi_j\left(x\right) y_{n+j} + h \sum_{j=0}^{m-1} \varphi_j\left(x\right) f\left(x_j, u\left(\bar{x}_j\right)\right), \ x_n \leq x \leq x_{n+k} \tag{3}$$

where $t$ denotes the number of interpolation point's $x_{n+1}, \ i = 0, 1, ..., t-1$ $m$ denotes the distinct collocation points $x_i \in \left[x_n, x_{n+k}\right] \ i = 0, 1, ..., m-1$.
The point's $x_i$ are chosen from the steps $x_{n+1}$ as well as one more off-step points. We make the following assumption:

(i) for a given mesh $x_n : x_n = a + nh, \ n = 0\left(1\right)N \ \ where; \ h = x_{n+1} - x_n$
N=$\left(b-a\right)/h \ \ is \ a \ cons\tan t \ \ step \ size$ ; and

(ii) that (1) has a unique solution and the coefficients $\phi_j\left(x\right) \ and \ \varphi_j\left(x\right)$ in (4) can be represented by polynomials of the form:

$$\phi_j\left(x\right) = \sum_{i=0}^{t+m-1} \phi_{j,i+1} x^i; j \in \{0, 1, \cdots, t-1\}$$

$$h\varphi_j(x) = \sum_{i=0}^{t+m-1} \varphi_{j,i+1} x^i; j \in \{0, 1, \cdots, m-1\} \tag{4}$$

with constant coefficients $\phi_{j,i+1},\ h\varphi_{j,i+1}$ to be determined using interpolation and collocation conditions:

$$U(x_{n+1}) = y_{n+1}, i \in \{0, 1, \cdots, t-1\}$$

$$U'(\bar{x}_j) = f(\bar{x}_j, u(\bar{x}_j)); j \in \{0, 1, \cdots, m-1\} \tag{5}$$

With these assumptions, we obtain a MC polynomial, see (Onumanyi et al, 1994), in the form:

$$U(x) = \sum_{i=0}^{t+m-1} a_i x^i, a_i = \sum_{j=0}^{i-1} c_{i+1,j+1} y_{n+j} + \sum_{j=0}^{m-1} c_{i+1,j+1} f_{n+j} \tag{6}$$

where $x_n \leq x \leq x_{n+k}$ *and* $c_{i,j}, i, j = 1, 2 ... t+m$are constants given by the elements of the inverse matrix C=D$^{-1}$. The multistep collocation matrix D is an m+1 square matrix of the type;

$$D = \begin{bmatrix} 1 & x_n & x_n^2 ... & x_n^{r+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 ... & x_{n+1}^{r+m-1} \\ . & . & . & . \\ . & . & . & . \\ 1 & x_{n+r-1} & x_{n+r-1}^2 ... & x_{n+r-1}^{r+m-1} \\ 0 & 1 & 2x_0 ... & (r+m-1)\, x_0^{r+m-2} \\ . & . & . & . \\ . & . & . & . \\ 0 & 1 & 2x_{m-1} ... & (r+m-1)\, x_{m-1}^{r+m-2} \end{bmatrix} \tag{7}$$

Is the multistep collocation matrix of dimension $(r+m)x(r+m)$. Then it follows from (7) that the column of matrix C=D$^{-1}$ give the continuous coefficients $\alpha_j(x)$ *and* $\beta_j(x)$

## 3. Derivation of the Continuous and Discrete Block Hybrid Methods:

Using the general multistep collocation methods see (Onumanyi., et al. 1994), (Yahaya and Mohammed, 2010) and (Mohammed, 2010) lead to the following D-matrix:

$$D=\begin{pmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 \\ 0 & 1 & 2x_{n+\mu} & 3x_{n+\mu}^2 & 4x_{n+\mu}^3 & 5x_{n+\mu}^4 \end{pmatrix} \quad (8)$$

using the general form of (3), this method will be expressed as::::::: help

(9) $$y(x) = \alpha_0 y_{n+2} + h\left[\beta_0 f_{n+2} + \beta_1 f_{n+3} + \beta_2 f_{n+4} + \beta_3 f_{n+5} + \beta_4 f_{n+\mu}\right]$$

With the help of the maple software package, it gives the column of $D^{-1}$**,** which are the elements of the matrix C. The elements of C are then used to generate the value of continuous coefficient

(10) $$\alpha_0(x), \quad \beta_0(x), \ \beta_1(x), \quad \beta_2(x), \quad \beta_3(x) \ \ and \ \beta_4(x)$$

The values of the continuous coefficient (10) are substituted into (9) to give the continuous form of the five step block methods as:

$$y(x) = y_{n+2} + 12(x-x_n)^5 - 15(\mu+12)(x-x_n)^4 h + 20(12\mu+47)(x-x_n)^3 h^2 - 30(47\mu+60)$$

$$(x-x_n)^2 h^3 + 3600\mu(x-x_n)h^4 - 2(1620\mu-1088)h^5\big]\, f_{n+2} \big/ 360(\mu-2)h^4$$

$$+\big[-12(x-x_n)^5 + 15(\mu+11)(x-x_n)^4 h - 20(11\mu+38)(x-x_n)^3 h^2 + 30(38\mu+40)(x-x_n)^2 h^3$$

$$-2400\mu(x-x_n)h^4 + 4(440\mu-244)h^5\big]\, f_{n+3} \big/ 120(\mu-3)h^4$$

$$+\big[12(x-x_n)^5 - 15(\mu+10)(x-x_n)^4 h + 20(10\mu+31)(x-x_n)^3 h^2 - 30(31\mu+30)(x-x_n)^2 h^3$$

$$+1800\mu(x-x_n)h^4 - 4(310\mu-164)h^5\big]\, f_{n+4} \big/ 120(\mu-4)h^4$$

$$+\big[-12(x-x_n)^5 + 15(\mu+9)(x-x_n)^4 h - 20(9\mu+26)(x-x_n)^3 h^2 + 30(26\mu+24)(x-x_n)^2 h^3$$

$$-1440\mu(x-x_n)h^4 + 4(240\mu-124)h^5\big]\, f_{n+5} \big/ 3650(\mu-4)h^4$$

(11)
$$\frac{+\big[6(x-x_n)^5 - 105(x-x_n)^4 h + 710(x-x_n)^3 h^2 - 2310(x-x_n)^2 h^3 + 3600(x-x_n)h^4 - 2152h^5\big]\, f_{n+\mu}}{30(\mu^4 + 120 - 154\mu + 71\mu^2 - 14\mu^3)h^4}$$

Evaluating (11) at the points:$x = x_{n+5}, x = x_{n+4}, x = x_{n+3}$ ,$x = x_{n+1}$, $x = x_n, x = x_{n+\frac{9}{2}}$ *and* $\mu = \frac{9}{2}$ to obtain the following six discrete hybrid methods which are used as a block integrator.

$$y_{n+5} - y_{n+2} = \frac{h}{200}\left[63 f_{n+2} + 285 f_{n+3} + 45 f_{n+4} + 192 f_{n+\frac{9}{2}} + 15 f_{n+5}\right]$$

$$y_{n+4} - y_{n+2} = \tfrac{h}{225}\left[71 f_{n+2} + 320 f_{n+3} + 15 f_{n+4} + 64 f_{n+\frac{9}{2}} - 20 f_{n+5}\right]$$

$$y_{n+3} - y_{n+2} = \tfrac{h}{1800}\left[599 f_{n+2} + 1805 f_{n+3} - 1515 f_{n+4} + 16064 f_{n+\frac{9}{2}} - 4345 f_{n+5}\right]$$

$$y_{n+1} - y_{n+2} = \tfrac{h}{1800}\left[-5129 f_{n+2} + 944 f_{n+3} - 17835 f_{n+4} + 16064 f_{n+\frac{9}{2}} - 4345 f_{n+5}\right]$$

$$y_{n+2} - y_{n} = \tfrac{h}{225}\left[3101 f_{n+2} - 8680 f_{n+3} + 18465 f_{n+4} - 17216 f_{n+\frac{9}{2}} + 4780 f_{n+5}\right]$$

(12)

$$y_{n+\frac{9}{2}} - y_{n+2} = \frac{h}{1155}\left[365 f_{n+2} + 1625 f_{n+3} + 375 f_{n+4} + 640 f_{n+\frac{9}{2}} - 125 f_{n+5}\right]$$

(12)

Equation (12) constitute the member of a zero-stable block integrators of order (5,5,5,5,5,5) and constants; $C_6 = \left[\frac{3}{400}, \frac{7}{900}, \frac{13}{1200}, \frac{811`}{3600}, \frac{721}{300}, \frac{25}{3072}\right]^T$ the application of the block integrators with $n = 0$ give the accurate values of $y_1,\ y_2,\ y_3,\ y_4$ along with $y_5$ as shown in table (5.1 and 5.2). To start the IVP integration on the sub-interval, we compute (12), when $n = 0$, i.e. the 1-block 5-point method to produce its unknown simultaneously without recourse to any starting method (predictor) to generate $y_1,\ y_2,\ y_3,\ y_4$ before computing $y_5$.

## 4. Convergence Analysis of Block Hybrid Methods:

Recall that, it is a desirable property for a numerical integrator to produce solution that behaves similar to the theoretical solution to a problem at all times. Thus several definitions, which call for the method to possess some "adequate" region of absolute stability, can be found in several literatures. See (Lambert, 1973), (Fatunla, 1992) and (Fatunla, 1994) etc. Following (Fatunla, 1992), the five integrator proposed in this report in equation (12) is put in the matrix-equation form and for easy analysis the result was normalized to obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix}$$

(13)

$$+h\begin{pmatrix} 0 & \frac{19679}{1800} & -\frac{11999}{360} & \frac{8659}{120} & -\frac{15208}{225} & \frac{6779}{360} \\ 0 & \frac{3101}{225} & -\frac{1736}{45} & \frac{1231}{15} & -\frac{17216}{225} & \frac{956}{45} \\ 0 & \frac{2823}{200} & -\frac{1503}{40} & \frac{3249}{40} & -\frac{1896}{25} & \frac{843}{40} \\ 0 & \frac{3172}{225} & -\frac{1672}{45} & \frac{1232}{15} & -\frac{17152}{225} & \frac{952}{45} \\ 0 & \frac{45117}{3200} & -\frac{23787}{640} & \frac{52731}{640} & -\frac{1899}{25} & \frac{13527}{640} \\ 0 & \frac{1015}{72} & -\frac{2675}{72} & \frac{1975}{24} & -\frac{680}{9} & \frac{1535}{72} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+\frac{9}{2}} \\ f_{n+5} \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

The first characteristic polynomial of the block hybrid method is given by:

$$\rho(R) = \det(RA^0 - A^1) \tag{14}$$

Substituting the value of $A^0$ and $A^1$ into the function above gives:

$$\begin{pmatrix} R & 0 & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & 0 & -1 \\ 0 & 0 & R & 0 & 0 & -1 \\ 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & R & 1 \\ 0 & 0 & 0 & 0 & 0 & R-1 \end{pmatrix} = [R^5(R-1)] \tag{15}$$

Therefore: $R_1 = R_2 = R_3 = R_4 = R_5 = 0$ *and* $R_6 = 1$. The hybrid method is zero stable and consistence, since the order of the method $R > 1$ and by (Henrici, 1962); the block hybrid method is convergent.

## 5. Numerical Experiment:

In this paper, we use newly constructed block hybrid methods and four step block hybrid Adams-mouton methods proposed by (Yahaya and Sokoto, 2010) to solve stiff and non-stiff initial value problems (IVP), in order to test for efficiency of the schemes derived.

**Example 5.1:**

Consider the initial value problem:

$$\begin{cases} y' = -y \ ; \ y(0) = 1 \\ 0 \le x \le 1 \ ; \ h = 0.1 \\ exact\ solution \ ; \ y(x) = e^{-x} \end{cases} \tag{16}$$

Firstly, we transform the schemes by substitution, to get a recurrence relation. Substituting $n = 0,\ 5,\ 10,\ \cdots$ and solving simultaneously using maple software package we obtain the required results displayed in tables below:

**Table for Example 5.1:**

| X | Exact Solution | Numerical Solution | | Absolute Error | |
|---|---|---|---|---|---|
| | | Proposed Hybrid Method for $k=5$ | Mohammed (2011) for $k=5$ | Proposed Hybrid Method for $k=5$ | Mohammed (2011) for $k=5$ |
| 0.1 | 0.9048374180 | 0.9048374180 | 0.9048549405 | 0.00000E+00 | 1.75225E-05 |
| 0.2 | 0.8187307531 | 0.8187307531 | 0.8187488967 | 0.00000E+00 | 1.81436E-05 |
| 0.3 | 0.7408182207 | 0.7408182205 | 0.7408344615 | 2.00000E-10 | 1.62408E-05 |
| 0.4 | 0.6703200460 | 0.6703200461 | 0.6703348438 | 1.00000E-10 | 1.47978E-05 |
| 0.5 | 0.6065306597 | 0.6065306603 | 0.6065438712 | 6.00000E-10 | 1.32115E-05 |
| 0.6 | 0.5488116361 | 0.5488116368 | 0.5488342186 | 7.00000E-10 | 2.25825E-05 |
| 0.7 | 0.4965853038 | 0.4965853042 | 0.4966071254 | 4.00000E-10 | 2.18216E-05 |
| 0.8 | 0.4493289641 | 0.4493289649 | 0.4493486023 | 8.00000E-10 | 1.96382E-05 |
| 0.9 | 0.4065696597 | 0.4065696606 | 0.4065874913 | 9.00000E-10 | 1.78316E-05 |
| 1.0 | 0.3678794412 | 0.3678794420 | 0.3678954677 | 8.00000E-10 | 1.60265E-05 |

**Example 5.2:**

Consider the initial value problem

$$\begin{cases} y' = -9y \ ; \ y(0) = e \\ 0 \leq x \leq 1 \ ; \ h = 0.1 \\ exact \ solution \ ; \ y(x) = e^{1-9x} \end{cases} \tag{17}$$

**Table for Example 5.2:**

| X | Exact Solution | Numerical Solution | | Absolute Error | |
|---|---|---|---|---|---|
| | | Proposed Hybrid Method for $k = 5$ | Mohammed (2011) for $k = 5$ | Proposed Hybrid Method for $k = 5$ | Mohammed (2011) for $k = 5$ |
| 0.1 | 1.10517E+00 | 1.10501E+00 | 1.252501337E+00 | 1.60000E-04 | 1.473304190E-01 |
| 0.2 | 4.49329E-01 | 4.49262E-01 | 5.267040462E-01 | 6.70000E-05 | 7.737508220E-02 |
| 0.3 | 1.82684E-01 | 1.82678E-01 | 2.125875480E-01 | 6.00000E-06 | 2.990402400E-02 |
| 0.4 | 7.42736E-02 | 7.42874E-02 | 8.737521120E-02 | 1.38000E-05 | 1.310163320E-02 |
| 0.5 | 3.01974E-02 | 3.01985E-02 | 3.381617705E-02 | 1.10000E-06 | 3.618794050E-03 |
| 0.6 | 1.22773E-02 | 1.22778E-02 | 1.558146272E-02 | 5.00000E-07 | 3.304122720E-03 |
| 0.7 | 4.99159E-03 | 4.99238E-03 | 6.552343872E-03 | 7.90000E-07 | 1.560749872E-03 |
| 0.8 | 2.02943E-03 | 2.03019E-03 | 2.644647840E-03 | 7.60000E-07 | 6.152168400E-04 |
| 0.9 | 8.25105E-04 | 8.25289E-04 | 1.086971770E-03 | 1.84000E07 | 2.618667700E-04 |
| 1.0 | 3.35463E-04 | 3.35538E-04 | 4.206825865E-04 | 7.50000E-08 | 8.713628650E-05 |

**Conclusions:**

A Collocation technique, which yields a method with continuous coefficients, is been presented for the approximate solution of first order ODEs with initial conditions. Two test examples of stiff and non-stiff problems have been solved to demonstrate the efficiency of the proposed method, and the results obtained compared favourably with the exact solution and block hybrid method of Mohammed (2011), which are all of step five (i.e. $k = 5$). Interestingly, all the discrete schemes used in the Block formulation were from a single continuous formulation (CF).

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