International Journal of Science and Technology Volume 4 No. 4, April, 2015

Error and Convergence Analysis of a Hybrid Runge- Kutta Type Method

[1] Muhammad R., [2]Y. A Yahaya, [3]A.S Abdulkareem
[1&2]Department of Mathematics/Statistics, Federal University of Technology, Minna.
[3]Department of Chemical Engineering, Federal University of Technology, Minna.

ABSTRACT

Implicit Runge- Kutta methods are used for solving stiff problems which mostly arise in real life problems. Convergence analysis helps us to determine an effective Runge- Kutta Method (RKM) to use, but due to the loss of linearity in Runge –Kutta Methods and the fact that the general Runge –Kutta Method makes no mention of the differential equation makes it impossible to define the order of the method independently of the differential equation. In this paper, we derived a hybrid Runge -Kutta Type method (RKTM) for $k = 1$, obtained the order and error constant and convergence analysis of the method.

Keywords: *Error, Convergence Analysis, Hybrid, Runge-Kutta Type method*

1. INTRODUCTION

The initial value problem for first order Ordinary Differential Equation is defined by

$$y' = f(x, y) \qquad y(x_0) = y_0 \qquad x \in [a, b] \qquad (1)$$

Butcher defined an s-stage Runge Kutta methods for the first order differential equation in the form

$$y_{n+1} = y_n + h\sum_{i,j=1}^{s} a_{ij}k_i \qquad (2)$$

where for $i = 1,2 \ldots \ldots \ldots . s$

$$k_i = f(x_i + \alpha_j h, y_n + h\sum_{i,j=1}^{s-1} a_{ij}k_j) \qquad (3)$$

The real parameters α_j, k_j, a_{ij} define the method. The method in Butcher array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & b^T \end{array}$$

Where $A = a_{ij} = \beta$

According to kulikov (2003) if the matrix A is strictly lower triangular (i:e the internal stages can be calculated without depending on later stages), then the method is called an explicit method, otherwise the internal stages depend not only on the previous stages but also on the current stage and later stages, then the method is called an Implicit method. This method is more suitable for solving stiff problems due to its high order of accuracy which makes it more superior to the explicit method.

2. DEFINITION OF TERMS

Definition 1

Order and Error Constant of Runge-Kutta Method

The **first** and **second** order Ordinary Differential Equation (ODE) methods are said to be of order $\boldsymbol{p}$ if $\boldsymbol{p}$ is the largest integer for which

$$y(x + h) - y(x) - h\varphi(x, y(x), h) = 0(h^{p+1}) \qquad (4)$$

$$y(x + h) - y(x) - h^2\varphi(x, y(x), y'(x), h^2) = 0(h^{p+2}) \qquad (5)$$

holds respectively. Where

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) \ldots \ldots \ldots \ldots + \frac{h^s}{s!}y^s(x) \qquad (6)$$

$\varphi(x, y(x), h) = y'(x+h) = f(x, y(x)),$ (7)

$\varphi(x, y(x), y'(x), h^2) = y''(x+h) = f(x, y(x), y'(x))$ (8)

in the taylor series expansion about x_0 and compare coefficients of $h^k y^k(x_0), y(x_0)$ is the interval value. The coefficient for which $\boldsymbol{p}$ is the largest integer is known as the **error constant**. (Adegboye 2013).

Definition 2

Consistency of Runge Kutta Methods

The first and second order Ordinary Differential Equation (ODE) methods are said to be consistent if

$\varphi(x, y(x), 0) \equiv f(x, y(x))$ (9)

$\varphi(x, y(x), y'(x), 0) \equiv f(x, y(x), y'(x))$ (10)

holds respectively.

Note that consistency demands that $\sum_1^s b_s = 1$, and $\sum_1^s b_s = \frac{1}{2}$ = for first and second order respectively. Also $\sum_1^s b_s$ is as shown in the butcher array table.

α	$\bar{A}$	A
	$\bar{b}^T$	b

$A = a_{ij} = \beta^2$ $\quad\bar{A} = \bar{a}_{ij} = \beta$ $\quad\beta = \beta e$

Definition 3

Convergence of Runge –Kutta Methods

If $f(x, y(x)), f(x, y(x), y'(x))$ represents first and second order respectively, then for such method consistency is necessary and sufficient for convergence. Hence the methods are said to be convergent if and only if they are consistent. (Adegboye 2013).

3. THE REFORMULATION OF RUNGE KUTTA TYPE METHOD FOR ORDER AND ERROR CONSTANT.

The initial value problem (IVP) for a system of first order Ordinary Differential Equation is defined by
$y' = f(x, y) \qquad y(x_0) = y \qquad x \in [a, b]$

The general s-stage Runge Kutta method is defined by

$y_{n+1} = y_n + h\sum_{i=1}^{s} a_{ij} k_i$ (11)

where for $i = 1,2 \ldots \ldots \ldots s$

$k_i = f(x_i + c_j h, y_n + h\sum_{i=1}^{s} a_{ij} k_j)$ (12)

The real parameters c_j, k_i, a_{ij} define the method. The method in Butcher array form can be written as

c	β
	W^T

Where $a_{ij} = \beta$

For $c_1, c_2, \ldots, c_s$ and $k_1, k_2, \ldots k_s$ in (12) we shall let $k_i = f_{c_i}$ implies $k_1 = f_{c_1}$, $k_2 = f_{c_2}$, $k_3 = f_{c_3}$ and $k_s = f_{c_s}$.

Consider the equation for the Block Hybrid Runge Kutta Type Backward Differentiation Formula for $K = 1$ given as

$$y_{n+\frac{1}{2}} = y_n + h(\tfrac{3}{4}k_2 - \tfrac{1}{4}k_3) \qquad (13a)$$

$$y_{n+1} = y_n + hk_2 \qquad (13b)$$

Where

$$k_1 = f(x_n, y_n) \qquad (14a)$$

$$k_2 = f(x_n + \tfrac{1}{2}h, y_n + h\left\{0k_1 + \tfrac{3}{4}k_2 - \tfrac{1}{4}k_3\right\}) \qquad (14b)$$

$$k_3 = f(x_n + h, y_n + h\{0k_1 + k_2 + 0k_3) \qquad (14c)$$

Since $k_i = f_{c_i}$, implies $k_1 = f_{c_1}$, $k_2 = f_{c_2}$, $k_3 = f_{c_3}$

Using equation (12), it implies $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = 1$.

Therefore $k_1 = f_0$, $k_2 = f_{\frac{1}{2}}$, $k_3 = f_1$, the equation now becomes

$$y_{n+\frac{1}{2}} = y_n + h(\tfrac{3}{4}f_{\frac{1}{2}} - \tfrac{1}{4}f_1) \qquad (15a)$$

$$y_{n+1} = y_n + hf_{\frac{1}{2}} \qquad (15b)$$

Taylor series expansion of

$$y_{n+\frac{1}{2}} = y\left(n + \frac{1}{2}h\right) = y(n) + \frac{1}{2}hy'(n) + \frac{\left(\frac{1}{2}h\right)^2}{2!}y''(n) + \frac{\left(\frac{1}{2}h\right)^3}{3!}y'''(n) + \cdots + \frac{\left(\frac{1}{2}h\right)^s}{s!}y^s(n)$$

$$y_{n+1} = y(n+h) = y(n) + hy'(n) + \frac{(h)^2}{2!}y''(n) + \frac{(h)^3}{3!}y'''(n) + \frac{(h)^4}{4!}y'^v(n) \ \ldots + \frac{(h)^s}{s!}y^s(n)$$

$$f_{\frac{1}{2}} = f\left(n + \frac{1}{2}h\right) = y'(n) + \frac{1}{2}hy''(n) + \frac{\left(\frac{1}{2}h\right)^2}{2!}y'''(n) + \frac{\left(\frac{1}{2}h\right)^3}{3!}y'^v(n) + \cdots + \frac{\left(\frac{1}{2}h\right)^{(s-1)}}{(s-1)!}y^s(n)$$

$$f_1 = f(n+h) = y'(n) + hy''(n) + \frac{(h)^2}{2!}y'''(n) + \frac{(h)^3}{3!}y'^v(n) + \cdots + \frac{(h)^{(s-1)}}{(s-1)!}y^s(n)$$

By substituting into the equation (15 a & b) above, we have

$y_{n+\frac{1}{2}} - y_n - h\left(\frac{3}{4}f_{\frac{1}{2}} - \frac{1}{4}f_1\right) = \frac{5}{96}h^3y^3$, the method is of order 2 and the error constant is $\frac{5}{96}$.

Also,

$y_{n+1} - y_n - hf_{\frac{1}{2}} = \frac{1}{24}h^3y^3$, the method is of order 2 and the error constant is $\frac{1}{24}$

From definition (2) and (3), the methods

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{3}{4} & \frac{-1}{4} \end{array}$$

1	0	1	0
	0	1	0

are consistent since $\sum_1^s b_s = 1$, hence convergent.

Consider this equation for the second derivative of $\boldsymbol{k = 1}$ given as

$$y_{n+\frac{1}{2}} = y_n + hy'_n + h^2\left(0k_1 + \frac{5}{16}k_2 - \frac{3}{16}k_3\right),$$

$$y'_{n+\frac{1}{2}} = y'_n + h\left(0k_1 + \frac{3}{4}k_2 - \frac{1}{4}k_3\right)$$

(16)

$$y_{n+1} = y_n + hy'_n + h^2\left(0k_1 + \frac{3}{4}k_2 - \frac{1}{4}k_3\right),$$

$$y'_{n+1} = y'_n + h(0k_1 + k_2 + 0k_3)$$

From equation (12) $c_1 = 0,\ c_2 = \frac{1}{2}, c_3 = 1$.

Therefore $k_1 = f_0, k_2 = f_{\frac{1}{2}}, k_3 = f_1$, the equation now becomes

$$y_{n+\frac{1}{2}} = y_n + hy'_n + h^2\left(0f_0 + \frac{5}{16}f_{\frac{1}{2}} - \frac{3}{16}f_1\right),$$

$$y'_{n+\frac{1}{2}} = y'_n + h\left(0f_0 + \frac{3}{4}f_{\frac{1}{2}} - \frac{1}{4}f_1\right)$$

$$y_{n+1} = y_n + hy'_n + h^2\left(0f_0 + \frac{3}{4}f_{\frac{1}{2}} - \frac{1}{4}f_1\right),$$

$$y'_{n+1} = y'_n + h\left(0f_0 + f_{\frac{1}{2}} + 0f_1\right)$$

The taylor series expansion of

$$f_{\frac{1}{2}} = f\left(n + \frac{1}{2}h\right) = y'' + \left(\frac{1}{2}h\right)y''' + \frac{(\frac{1}{2}h)^2}{2!}y^{iv} + \cdots \ldots\ldots.. + \frac{(\frac{1}{2}h)^s}{s!}y^{s+2}$$

$$f_1 = f(n+h) = y'' + hy''' + \frac{(h)^2}{2!}y^{iv} + \ldots\ldots\ldots\ldots + \frac{(h)^s}{s!}y^{s+2}$$

Substituting the values in the above equation, we obtained the Order and Error Constant for the second derivative of $\boldsymbol{k = 1}$ of the Block Hybrid Runge Kutta Type method as tabulated below.

Method	Order	Error Constant
$y_{n+\frac{1}{2}} = y_n + hy'_n + h^2\left(0k_1 + \frac{5}{16}k_2 - \frac{3}{16}k_3\right)$	2	$\frac{5}{96}$
$y_{n+1} = y_n + hy'_n + h^2\left(0k_1 + \frac{3}{4}k_2 - \frac{1}{4}k_3\right)$	2	$\frac{1}{24}$

From definition (2) and (3), the methods

$$\begin{array}{c|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{5}{16} & \frac{-3}{16} \\
1 & 0 & 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{4} \\
\hline
 & 0 & 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{4}
\end{array}$$

are consistent since $\sum_1^s b_s = \frac{1}{2}$, hence convergent.

4. CONCLUSION

The procedure adopted speeds up computation and reduces computational effort in carrying out the convergence analysis of the block hybrid Runge Kutta Type Method (RKTM).The derivation is done only once which allows higher order to be formulated.

Acknowledgements

We wish to express our profound gratitude to the Almighty God Who makes things all possible. Our appreciation goes to all our colleagues in Mathematics/Statistic department and Chemical Engineering Department for peaceful coexistence and providing an enabling environment suitable for undertaking a research work. To all whose work we have found indispensable in the course of this research. All such have been duly acknowledged.

REFERENCES

[1] Butcher, J.C (2008). *Numerical methods for ordinary differential equations.* John Wiley & Sons.

[2] Butcher, J.C & Hojjati, G. (2005). Second derivative methods with runge-kutta stability. *Numerical Algorithms*, 40, 415-429.

[3] Kulikov, G. Yu. (2003). “Symmetric Runge Kutta Method and their stability”. *Russ J. Numeric Analyze and Maths Modelling. 18(1): 13-41*

[4] Yahaya, Y.A. & Adegboye, Z.A. (2011). Reformulation of quade’s type four-step block hybrid multstep method into runge-kutta method for solution of first and second order ordinary differential equations. *Abacus, 38(2)*, 114-124.

[5] Yahaya Y.A. and Adegboye Z.A. (2013). Derivation of an implicit six stage block runge kutta type method for direct integration of boundary value problems in second order ode using the quade type multistep method. *Abacus*, *40(2)*, 123-132.