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Reformulation of Block Implicit Linear Multistep Method into Runge Kutta Type Method for Initial Value Problem

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ABSTRACT

In this research work, we reformulated the block hybrid Backward Differentiation Formula (BDF) for $k = 4$ into Runge Kutta Type Method (RKTM) of the same step number for the solution of Initial value problem in Ordinary Differential Equation (ODE). The method can be use to solve both first and second order (special or general form). It can also be extended to solve higher order ODE. This method differs from conventional BDF as derivation is done only once.

Keywords: *Block, Implicit, Runge-Kutta Type, Reformulation.*

1. INTRODUCTION

In an earlier work (Muhammad and Yahaya (2011)), the idea of multistep collocation was adopted to obtain the continous forms of some derived hybrid Backward Differentiation Formulae (BDF) collated from literature. The continous forms were evaluated at some grid and off grid points which gave rise to block BDF (Hybrid and Non-hybrid) for $k = 1,2,3 \ldots .6$. The Block Backward Differentiation Formulae (Hybrid and Non-hybrid) were used to obtain solutions of some stiff and non-stiff Initial Value Problems.

In this paper, we seek to reformulate the Block Backward Differentiation Formulae (Hybrid and Non-hybrid) for $k = 4$ into Runge Kutta Type Method for the solution of Initial Value Problems in Ordinary Differential Equations (ODE) of the form

$$y' = f(x,y) \qquad y(x_0) = y \qquad (1)$$

$$y'' = f(x,y) \qquad y(x_0) = y \quad y'(x_0) = \beta \qquad (2)$$

$$y'' = f(x,y,y') \qquad y(x_0) = y \quad y'(x_0) = \beta \qquad (3)$$

We consider the numerical solution of the Initial Value Problem that has benefits such as self starting, high order, low error constants, satisfactory stability property such as A-stability and low implementation cost. We emphasize the combination of multistep structure with the use of off grid points and seek a method that is both multistage and multivalue. This will enable us to extend the general linear formulation to the high order Runge kutta case (Butcher 2003) by considering a polynomial

$$y(x) = \sum_{j=1}^{t-1} \alpha_j(x)\, y_{n+j} + h \sum_{j=1}^{m-1} \beta_j\, f(\bar{x}_j, y(\bar{x}_j)) \qquad (4)$$

Where t denotes the number of interpolation points $x_{n+j}, j = 0,1 \ldots .t-1$ and m denotes the distinct collocation points $\bar{x}_j \epsilon [x_n, x_{n+k}], j = 0,1 \ldots \ldots m-1$ chosen from the given step $[x_n, x_{n+k}]$.

Butcher defined an s-stage Runge Kutta method for the first order differential equation in the form

$$y_{n+1} = y_n + h \sum_{i,j=1}^{s} a_{ij} k_i \qquad (5)$$

where for $i = 1,2 \ldots \ldots \ldots .s$

$$k_i = f(x_i + \alpha_j h, y_n + h \sum_{i,j=1}^{s-1} a_{ij} k_j) \qquad (6)$$

The real parameters α_j, k_i, a_{ij} define the method. The method in Butcher array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & b^T \end{array}$$

Where $a_{ij} = \beta$

The Runge Kutta Nystrom (RKN) method is an extension of Runge Kutta method for second order ODE of the form

$$y'' = f(x, y, y') \qquad y(x_0) = y_0 \quad y'(x_0) = y_0' \tag{7}$$

An S-stage implicit Runge Kutta Nystrom for direct integration of second order initial value problem is defined in the form

$$y_{n+1} = y_n + \alpha_i h y_n' + h^2 \sum_{i,j=1}^{s} a_{ij} k_j \tag{8a}$$

$$y_{n+1}' = y_n' + h \sum_{i,j=1}^{s} \bar{a}_{ij}\, k_j \tag{8b}$$

where for $i = 1,2 \ldots \ldots \ldots s$

$$k_i = f\left(x_i + \alpha_j h, y_n + \alpha_i h y_n' + h^2 \sum_{i,j=1}^{s} a_{ij} k_j\,, y_n' + h \sum_{i,j=1}^{s} \bar{a}_{ij}\, k_j\right) \tag{8c}$$

The real parameters $\alpha_j, k_j, a_{ij}, \bar{a}_{ij}$ define the method, the method in butcher array form is expressed as

$$\begin{array}{c|c|c} \alpha & \bar{A} & A \\ \hline & \bar{b}^T & b \end{array}$$

$A = a_{ij} = \beta^2 \qquad \bar{A} = \bar{a}_{ij} = \beta \qquad \beta = \beta e$

2. CONSTRUCTION OF THE BLOCK HYBRID BACKWARD DIFFERENTIATION FORMULA WHEN $K = 4$ (BHBDF4)

Consider the approximate solution to (1) in the form of power series

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j \tag{9}$$

$\alpha \in R, j = 0(1)t + m - 1, y \in C^m(a, b) \subset P(x)$

$$y'(x) = \sum_{j=0}^{t+m-1} j\alpha_j x^{j-1} \tag{10}$$

Where α_j's are the parameters to be determined, t and m are the points of interpolation and collocation respectively.

When $K = 4$, we interpolate $(t = 5)$ at $j = 0, \frac{1}{2}, 1,2,3$ and collocate $(m = 1)$ at $j = 4$. The equation can be expressed as

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j = y_{n+j} \qquad j = 0, \tfrac{1}{2}, 1,2,3 \tag{11}$$

$$y'(x) = \sum_{j=0}^{t+m-1} j\alpha_j x^{j-1} = f_{n+j} \qquad j = 4 \tag{12}$$

The general form of the proposed method upon addition of one off grid point is expressed as;

$$\bar{y}(x) = \alpha_1(x) y_n + \alpha_2(x) y_{n+1} + \alpha_3(x) y_{n+2} + \alpha_4(x) y_{n+3} + \alpha_5(x) y_{n+\frac{1}{2}} + h\beta_0(x) f_{n+4} \tag{13}$$

The matrix D of dimension $(t + m) * (t + m)$ of the proposed method is expressed as:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 & (x_n + h)^5 \\ 1 & x_n + 2h & (x_n + 2h)^2 & (x_n + 2h)^3 & (x_n + 2h)^4 & (x_n + 2h)^5 \\ 1 & x_n + 3h & (x_n + 3h)^2 & (x_n + 3h)^3 & (x_n + 3h)^4 & (x_n + 3h)^5 \\ 1 & x_n + \frac{1}{2}h & (x_n + \frac{1}{2}h)^2 & (x_n + \frac{1}{2}h)^3 & (x_n + \frac{1}{2}h)^4 & (x_n + \frac{1}{2}h)^5 \\ 0 & 1 & 2x_n + 8h & 3(x_n + 4h)^2 & 4(x_n + 4h)^3 & 5(x_n + 4h)^4 \end{bmatrix}$$

Using the maple software package, we invert the matrix D, to obtain columns which form the matrix C. The elements of C are used to generate the continous coefficients of the method as:

$$
\begin{aligned}
\alpha_1(x) &= C_{11} + C_{21}x + C_{31}x^2 + C_{41}x^3 + C_{51}x^4 + C_{61}x^5 \\
\alpha_2(x) &= C_{12} + C_{22}x + C_{32}x^2 + C_{42}x^3 + C_{52}x^4 + C_{62}x^5 \\
\alpha_3(x) &= C_{13} + C_{23}x + C_{33}x^2 + C_{43}x^3 + C_{53}x^4 + C_{63}x^5 \qquad (14) \\
\alpha_4(x) &= C_{14} + C_{24}x + C_{34}x^2 + C_{44}x^3 + C_{54}x^4 + C_{64}x^5 \\
\alpha_5(x) &= C_{15} + C_{25}x + C_{35}x^2 + C_{45}x^3 + C_{55}x^4 + C_{65}x^5 \\
\beta_0(x) &= C_{16} + C_{26}x + C_{36}x^2 + C_{46}x^3 + C_{56}x^4 + C_{66}x^5
\end{aligned}
$$

The values of the continous coefficients (14) are substituted into (13) to give as the continous form of the four step block hybrid BDF with one off step interpolation point.

$$
\begin{aligned}
y(x) = {} & \Big(\frac{1}{2388}\frac{1}{h^5}(2388h^5 + 9688h^4 + 13191x_n^2h^3 + 7666x_n^3h^2 + 1953x_n^4h + 178x_n^5) - \frac{1}{1194}\frac{1}{h^5}(4844h^4 + 13191x_nh^3 + \\
& 11499x_n^2h^2 + 3906x_n^3h + 445x_n^4)x + \frac{1}{2388}\frac{13191h^3+22998x_nh^2+11718x_n^2h+1780x_n^3}{h^5}x^2 - \frac{1}{1194}\frac{3833h^2+3906x_nh+890x_n^2}{h^5}x^3 + \\
& \frac{1}{2388}\frac{1953h+890x_n}{h^5}x^4 - \frac{89}{1194h^5}x^5\Big)y_n + \Big(\frac{1}{398}\frac{1}{h^5}\big(x_n(1534h^4 + 4694x_nh^3 + 3785x_n^2h^2 + 1139x_n^3h + 114x_n^4)\big) - \\
& \frac{1}{398}\frac{1}{h^5}\big(x_n(1536h^4 + 4694x_nh^3 + 3785x_n^2h^2 + 1139x_n^3h + 114x_n^4)\big)x + \frac{1}{398}\frac{4694h^3+11355x_nh^2+6834x_n^2h+1140x_n^3}{h^5}x^2 - \\
& \frac{1}{398}\frac{3785h^2+4556x_nh+1140x_n^2}{h^5}x^3 + \frac{1}{398}\frac{1139h+570x_n}{h^5}x^4 - \frac{57}{199h^5}x^5\Big)y_{n+1} + \\
& \Big(-\frac{1}{2388}\frac{1}{h^5}\big(x_n(7591x_nh^3 + 7978x_n^2h^2 + 2837x_n^3h + 314x_n^4 + 2136h^4)\big) \\
& \quad + \frac{1}{1194}\frac{1}{h^5}(7591x_nh^3 + 11967x_n^2h^2 + 5674x_n^3h + 785x_n^4 + 1068h^4)x \\
& \quad - \frac{1}{2388}\frac{7591h^3 + 23934x_nh^2 + 17022x_n^2h + 3140x_n^3}{h^5}x^2 + \frac{1}{1194}\frac{3989h^2 + 5674x_nh + 1570x_n^2}{h^5}x^3 \\
& \quad - \frac{1}{2388}\frac{2837h + 1570x_n}{h^5}x^4 + \frac{157}{1194h^5}x^5\Big)y_{n+2} \\
& + \Big(\frac{1}{5970}\frac{1}{h^5}\big(x_n(4613x_n^2h^2 + 1893x_n^3h + 230x_n^4 + 4038x_nh^3 + 1088h^4)\big) \\
& \quad - \frac{1}{5970}\frac{1}{h^5}(13839x_n^2h^2 + 7572x_n^3h + 1150x_n^4 + 785x_n^4 + 8076x_nh^3 + 1088h^4)x \\
& \quad + \frac{1}{5970}\frac{13839x_nh^2 + 11358x_n^2h + 2300x_n^3 + 4038h^3}{h^5}x^2 - \frac{1}{5970}\frac{4613h^2 + 7572x_nh + 2300x_n^2}{h^5}x^3 \\
& \quad + \frac{1}{5970}\frac{1893h + 1150x_n}{h^5}x^4 + -\frac{23}{597h^5}x^5\Big)y_{n+3} \\
& \Big(-\frac{32}{2985}\frac{1}{h^5}\big(x_n(25x_n^4 + 262x_n^3h + 947x_n^2h^2 + 1382x_nh^3 + 672h^4)\big) + \frac{32}{2985}\frac{1}{h^5}(125x_n^4 + 1048x_n^3h + 2841x_n^2h^2 + 2764x_nh^3 + \\
& 672h^4)x - \frac{32}{2985}\frac{1}{h^5}(250x_n^3 + 1572x_n^2h + 2841x_nh^2 + 1382h^3)x^2 + \frac{32}{2985}\frac{1}{h^5}(250x_n^2 + 1048x_nh + 947h^2)x^3 - \frac{32}{2985}\frac{1}{h^5}(125x_n + \\
& 262h)x^4 + \frac{160}{597h^5}x^5)y_{n+1/2} + \frac{1}{h^4}\Big(-\frac{1}{398}x_n(2x_n^4 + 13x_n^3h + 28x_n^2h^2 + 23x_nh^3 + 6h^4) + \Big(\frac{1}{199}(5x_n^4 + 26x_n^3h + 42x_n^2h^2 + \\
& 23x_nh^3 + 3h^4)\Big)x - \Big(\frac{1}{398}(20x_n^3 + 78x_n^2h + 84x_nh + 23h^3)\Big)x^2 + (\frac{2}{199}(5x_n^2 + 13x_nh + 7h^2)x^3 - \Big(\frac{1}{398}(10x_n + 13h)x^4\Big) + \\
& \frac{1}{199}x^5\Big)f_{n+4} \qquad (15)
\end{aligned}
$$

Evaluating (15) at point $x = x_{n+4}$ and its derivative at $x = x_{n+3}, x = x_{n+1,}\ , x = x_{n+2}\ , x = x_{n+1/2}$ yields the following five discrete hybrid schemes which are used as a block integrator;

$$-\frac{784}{199}y_{n+1} + \frac{588}{199}y_{n+2} - \frac{2352}{995}y_{n+3} + y_{n+4} + \frac{3072}{995}y_{n+1/2} = \frac{147}{199}y_n + \frac{84}{199}hf_{n+4}$$

$$-y_{n+1} - \frac{925}{2415}y_{n+2} + \frac{143}{2415}y_{n+3} + \frac{3712}{2415}y_{n+1/2} = \frac{515}{2415}y_n - \frac{1990}{2415}hf_{n+1} + \frac{10}{2415}hf_{n+4}$$

$$\frac{6390}{2305}y_{n+1} - y_{n+2} - \frac{942}{2305}y_{n+3} - \frac{3968}{2305}y_{n+1/2} = -\frac{825}{2305}y_n - \frac{2985}{2305}hf_{n+2} - \frac{45}{2305}hf_{n+4}$$

$$-\frac{19125}{9883}y_{n+1} + \frac{18075}{9883}y_{n+2} - y_{n+3} + \frac{14208}{9883}y_{n+1/2} = \frac{3275}{9883}y_n - \frac{5970}{9883}hf_{n+3} + \quad \frac{450}{9883}hf_{n+4} \qquad (16)$$

$$-\frac{204750}{125888}y_{n+1} + \frac{31675}{125888}y_{n+2} - \frac{5838}{125888}y_{n+3} + y_{n+\frac{1}{2}} \quad = -\frac{53025}{125888}y_n - \frac{95520}{125888}hf_{n+\frac{1}{2}} - \quad \frac{450}{125888}hf_{n+4}$$

It is of order $[5,5,5,5,5]^T$ with error constants $[-\frac{49}{995}, -\frac{227}{47760}, \frac{241}{23880}, -\frac{283}{9552}, \frac{1561}{305640}]^T$

Re-arranging the block implicit hybrid schemes simultaneously we obtained the following block scheme

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{28800}\left\{30976f_{n+\frac{1}{2}} - 24035f_{n+1} + 10985f_{n+2} - 4281f_{n+3} + 755f_{n+4}\right\}$$

$$y_{n+1} = y_n + \frac{h}{12600}\left\{16064f_{n+\frac{1}{2}} - 6265f_{n+1} + 4165f_{n+2} - 1659f_{n+3} + 295f_{n+4}\right\}$$

(17)

$$y_{n+2} = y_n + \frac{h}{1575}\left\{1856f_{n+\frac{1}{2}} + 140f_{n+1} + 1435f_{n+2} - 336f_{n+3} + 55f_{n+4}\right\}$$

$$y_{n+3} = y_n + \frac{h}{1400}\left\{1728f_{n+\frac{1}{2}} - 105f_{n+1} + 2205f_{n+2} + 357f_{n+3} + 15f_{n+4}\right\}$$

$$y_{n+4} = y_n + \frac{h}{225}\left\{256f_{n+\frac{1}{2}} + 40f_{n+1} + 260f_{n+2} + 264f_{n+3} + 80f_{n+4}\right\}$$

3. REFORMULATION OF THE BHBDF4 INTO RUNGE KUTTA TYPE METHOD (RKTM)

Reformulating the block hybrid method with the coefficients as characterized by the butcher array

$$\begin{array}{c|c} \alpha & \beta \\ \hline & b^T \end{array}$$

Where $a_{ij} = \beta$

Gives

$$\begin{array}{c|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{242}{225} & -\frac{4807}{5760} & \frac{2197}{5760} & -\frac{1427}{9600} & \frac{151}{5760} \\ 4 & 0 & \frac{256}{225} & \frac{8}{45} & \frac{52}{45} & \frac{88}{75} & \frac{16}{45} \\ 3 & 0 & \frac{216}{175} & -\frac{3}{40} & \frac{63}{40} & \frac{51}{200} & \frac{3}{280} \end{array}$$

2	0	$\frac{1856}{1575}$	$\frac{4}{45}$	$\frac{41}{45}$	$-\frac{16}{75}$	$\frac{11}{315}$
1	0	$\frac{2008}{1575}$	$\frac{8}{45}$	$\frac{52}{45}$	$\frac{88}{75}$	$\frac{16}{45}$
	0	$\frac{2008}{1575}$	$\frac{8}{45}$	$\frac{52}{45}$	$\frac{88}{75}$	$\frac{16}{45}$

Using Equation (5) we obtained an implicit 6-stage block Runge kutta type method of uniform order 4.

$$y_{n+\frac{1}{2}} = y_n + h(0k_1 + \frac{242}{225}k_2 - \frac{4807}{5760}k_3 + \frac{2197}{5760}k_4 - \frac{1427}{9600}k_5 + \frac{151}{5760}k_6)$$

$$y_{n+1} = y_n + h\left(0k_1 + \frac{2008}{1575}k_2 - \frac{179}{360}k_3 + \frac{119}{360}k_4 - \frac{79}{600}k_5 + \frac{59}{2520}k_6\right)$$

(18)

$$y_{n+2} = y_n + h\left(0k_1 + \frac{1856}{1575}k_2 + \frac{4}{45}k_3 + \frac{41}{45}k_4 - \frac{16}{75}k_5 + \frac{11}{315}k_6\right)$$

$$y_{n+3} = y_n + h\left(0k_1 + \frac{216}{175}k_2 - \frac{3}{40}k_3 + \frac{63}{40}k_4 + \frac{51}{200}k_5 + \frac{3}{280}k_6\right)$$

$$y_{n+4} = y_n + h\left(0k_1 + \frac{256}{225}k_2 + \frac{8}{45}k_3 + \frac{52}{45}k_4 + \frac{88}{75}k_5 + \frac{16}{45}k_6\right)$$

Where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{2}h, y_n + h\left(0k_1 + \frac{242}{225}k_2 - \frac{4807}{5760}k_3 + \frac{2197}{5760}k_4 - \frac{1427}{9600}k_5 + \frac{151}{5760}k_6\right))$$

$$k_3 = f(x_n + h, y_n + h\left(0k_1 + \frac{2008}{1575}k_2 - \frac{179}{360}k_3 + \frac{119}{360}k_4 - \frac{79}{600}k_5 + \frac{59}{2520}k_6\right))$$

$$k_4 = f(x_n + 2h, y_n + h\left(0k_1 + \frac{1856}{1575}k_2 + \frac{4}{45}k_3 + \frac{41}{45}k_4 - \frac{16}{75}k_5 + \frac{11}{315}k_6\right))$$

$$k_5 = f(x_n + 3h, y_n + h\left(0k_1 + \frac{216}{175}k_2 - \frac{3}{40}k_3 + \frac{63}{40}k_4 + \frac{51}{200}k_5 + \frac{3}{280}k_6\right))$$

$$k_6 = f(x_n + 4h, y_n + h\left(0k_1 + \frac{256}{225}k_2 + \frac{8}{45}k_3 + \frac{52}{45}k_4 + \frac{88}{75}k_5 + \frac{16}{45}k_6\right))$$

Extending the method (18) with the coefficient as shown in the butcher array

α	$\bar{A}$	A
	$\bar{b}^T$	b

$A = a_{ij} = \beta^2$ $\qquad$ $\bar{A} = \bar{a}_{ij} = \beta$ $\qquad$ $\beta = \beta e$

0	0	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{242}{225}$	$-\frac{4807}{5760}$	$\frac{2197}{5760}$	$-\frac{1427}{9600}$	$\frac{151}{5760}$	0	$\frac{583}{1500}$	$-\frac{24937}{57600}$	$\frac{5339}{19200}$	$-\frac{13297}{96000}$	$\frac{1711}{57600}$
4	0	$\frac{256}{225}$	$\frac{8}{45}$	$\frac{52}{45}$	$\frac{88}{75}$	$\frac{16}{45}$	0	$\frac{5248}{1125}$	$-\frac{24}{25}$	$\frac{856}{225}$	$\frac{104}{375}$	$\frac{16}{75}$
3	0	$\frac{216}{175}$	$-\frac{3}{40}$	$\frac{63}{40}$	$\frac{51}{200}$	$\frac{3}{280}$	0	$\frac{2988}{875}$	$-\frac{87}{100}$	$\frac{459}{200}$	$-\frac{54}{125}$	$\frac{129}{1400}$
2	0	$\frac{1856}{1575}$	$\frac{4}{45}$	$\frac{41}{45}$	$-\frac{16}{75}$	$\frac{11}{315}$	0	$\frac{1952}{875}$	$-\frac{208}{225}$	$\frac{76}{75}$	$-\frac{148}{375}$	$\frac{118}{1575}$
1	0	$\frac{2008}{1575}$	$\frac{8}{45}$	$\frac{52}{45}$	$\frac{88}{75}$	$\frac{16}{45}$	0	$\frac{7804}{7875}$	$-\frac{58}{75}$	$\frac{797}{1800}$	$-\frac{301}{1500}$	$\frac{169}{4200}$
	0	$\frac{2008}{1575}$	$\frac{8}{45}$	$\frac{52}{45}$	$\frac{88}{75}$	$\frac{16}{45}$	0	$\frac{7804}{7875}$	$-\frac{58}{75}$	$\frac{797}{1800}$	$-\frac{301}{1500}$	$\frac{169}{4200}$

NOTE:

The butcher table is being rearranged with the off grid points appearing first, followed by the $c_{i's}$ in descending order. This is done in other to satisfy the consistency condition.

Using Equation (8), we obtained an implicit 6 stage block Runge kutta Type method of uniform order 5.

$$y_{n+\frac{1}{2}} = y_n + \tfrac{1}{2}hy'_n + h^2\left(0k_1 + \tfrac{583}{1500}k_2 - \tfrac{24937}{57600}k_3 + \tfrac{5339}{19200}k_4 - \tfrac{13297}{96000}k_5 + \tfrac{1711}{57600}k_6\right),$$

$$y'_{n+\frac{1}{2}} = y'_n + h\left(0k_1 + \tfrac{242}{225}k_2 - \tfrac{4807}{5760}k_3 + \tfrac{2197}{5760}k_4 - \tfrac{1427}{9600}k_5 + \tfrac{151}{5760}k_6\right)$$

$$y_{n+1} = y_n + hy'_n + h^2\left(0k_1 + \frac{7804}{7875}k_2 - \frac{58}{75}k_3 + \frac{797}{1800}k_4 - \frac{301}{1500}k_5 + \frac{169}{4200}k_6\right),$$

$$y'_{n+1} = y'_n + h\left(0k_1 + \tfrac{2008}{1575}k_2 - \tfrac{179}{360}k_3 + \tfrac{119}{360}k_4 - \tfrac{79}{600}k_5 + \tfrac{59}{2520}k_6\right)$$

$$y_{n+2} = y_n + 2hy'_n + h^2\left(0k_1 + \frac{1952}{875}k_2 - \frac{208}{225}k_3 + \frac{76}{75}k_4 - \frac{148}{375}k_5 + \frac{118}{1575}k_6\right),$$

$$y'_{n+2} = y'_n + h\left(0k_1 + \tfrac{1856}{1575}k_2 + \tfrac{4}{45}k_3 + \tfrac{41}{45}k_4 - \tfrac{16}{75}k_5 + \tfrac{11}{315}k_6\right)$$

(21)

$$y_{n+3} = y_n + 3hy'_n + h^2\left(0k_1 + \frac{2988}{875}k_2 - \frac{87}{100}k_3 + \frac{459}{200}k_4 - \frac{54}{125}k_5 + \frac{129}{1400}k_6\right),$$

$$y'_{n+3} = y'_n + h\left(0k_1 + \tfrac{216}{175}k_2 - \tfrac{3}{40}k_3 + \tfrac{63}{40}k_4 + \tfrac{51}{200}k_5 + \tfrac{3}{280}k_6\right)$$

$$y_{n+4} = y_n + 4hy'_n + h^2\left(0k_1 + \frac{5248}{1125}k_2 - \frac{24}{25}k_3 + \frac{856}{225}k_4 + \frac{104}{375}k_5 + \frac{16}{75}k_6\right),$$

$$y'_{n+4} = y'_n + h\left(0k_1 + \frac{256}{225}k_2 + \frac{8}{45}k_3 + \frac{52}{45}k_4 + \frac{88}{75}k_5 + \frac{16}{45}k_6\right)$$

where

$$k_1 = f(x_n, y_n, y'_n)$$

$$k_2 = f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hy'_n + h^2\left(0k_1 + \tfrac{583}{1500}k_2 - \tfrac{24937}{57600}k_3 + \tfrac{5339}{19200}k_4 - \tfrac{13297}{96000}k_5 + \tfrac{1711}{57600}k_6\right),$$

$$y'_{n+\frac{1}{2}} = y'_n + h\left(0k_1 + \tfrac{242}{225}k_2 - \tfrac{4807}{5760}k_3 + \tfrac{2197}{5760}k_4 - \tfrac{1427}{9600}k_5 + \tfrac{151}{5760}k_6\right))$$

$$k_3 = f(x_n + h, y_n + hy'_n + h^2\left(0k_1 + \tfrac{7804}{7875}k_2 - \tfrac{58}{75}k_3 + \tfrac{797}{1800}k_4 - \tfrac{301}{1500}k_5 + \tfrac{169}{4200}k_6\right),$$

$$y'_{n+1} = y'_n + h\left(0k_1 + \tfrac{2008}{1575}k_2 - \tfrac{179}{360}k_3 + \tfrac{119}{360}k_4 - \tfrac{79}{600}k_5 + \tfrac{59}{2520}k_6\right))$$

$$k_4 = f(x_n + 2h, y_n + 2hy'_n + h^2\left(0k_1 + \tfrac{1952}{875}k_2 - \tfrac{208}{225}k_3 + \tfrac{76}{75}k_4 - \tfrac{148}{375}k_5 + \tfrac{118}{1575}k_6\right),$$

$$y'_n + h\left(0k_1 + \tfrac{1856}{1575}k_2 + \tfrac{4}{45}k_3 + \tfrac{41}{45}k_4 - \tfrac{16}{75}k_5 + \tfrac{11}{315}k_6\right))$$

$$k_5 = f(x_n + 3h, y_n + 3hy'_n + h^2\left(0k_1 + \tfrac{2988}{875}k_2 - \tfrac{87}{100}k_3 + \tfrac{459}{200}k_4 - \tfrac{54}{125}k_5 + \tfrac{129}{1400}k_6\right),$$

$$y'_n + h\left(0k_1 + \tfrac{216}{175}k_2 - \tfrac{3}{40}k_3 + \tfrac{63}{40}k_4 + \tfrac{51}{200}k_5 + \tfrac{3}{280}k_6\right))$$

$$k_6 = f(x_n + 4h, y_n + 4hy'_n + h^2\left(0k_1 + \frac{5248}{1125}k_2 - \frac{24}{25}k_3 + \frac{856}{225}k_4 + \frac{104}{375}k_5 + \frac{16}{75}k_6\right),$$

$$y'_n + h\left(0k_1 + \tfrac{256}{225}k_2 + \tfrac{8}{45}k_3 + \tfrac{52}{45}k_4 + \tfrac{88}{75}k_5 + \tfrac{16}{45}k_6\right))$$

4. NUMERICAL EXAMPLES

To study the efficiency of the method we present some numerical examples widely used by reknown authors such as Yahaya and Adegboye (2013), Sunday etal (2013).

Example 1: (SIR Model)

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of model derives from the fact that they involve coupled equations relating the number of susceptible people $S(t)$, the number of infected $I(t)$, and the number of people who have recovered $R(t)$. This is a good and simple model for many infectious diseases including measles, mumps and rubella. It is given by the following three coupled equations

$$\frac{dS}{dt} = \mu(1 - S) - \beta IS$$

$$\frac{dI}{dt} = -\mu I - \gamma I + \beta IS$$

$$\frac{dR}{dt} = -\mu R + \gamma I$$

Where μ , γ and β are positive parameters. Define y to be

$$y = S + I + R$$

And adding equation (1), (2) and (3) we obtain the following evolution equation for y

$$y' = \mu(1 - y)$$

Taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population), we obtain

$$y' = 0.5(1 - y), y(0) = 0.5$$

Exact Solution

$$y(t) = 1 - 0.5e^{-0.5t}$$

Applying the Runge Kutta Type Method (RKTM) to this problem yields the following result as indicated in table 1.

Problem 2

$$y'' - xy' + 4y = 0, \quad y(0) = 3, \quad y'(0) = 0, \quad h = 0.1 \qquad 0.1 < x < 0.4$$

Exact Solution

$$y(x) = x^4 6x^2 + 3$$

Applying the Runge Kutta Type Method (RKTM) to this problem yields the following result as indicated in table 2

TABLE 1: PERFORMANCE OF RKTM ON PROBLEM 1

t	Exact Solution	Computed Solution	Error
0.1	0.524385287749	0.5243852876	1.49E − 10
0.2	0.547581290982	0.5475812908	1.82E − 10
0.3	0.569646011787	0.5696460117	8.70001E − 11
0.4	0.590634623461	0.5906346233	1.61E − 10
0.5	0.610599608464	0.6105996083	1.64E − 10
0.6	0.629590889659	0.6295908895	1.59E − 10
0.7	0.647655955141	0.6476559550	1.41E − 10
0.8	0.664839976982	0.6648399767	2.82E − 10
0.9	0.681185924189	0.6811859239	2.89E − 10
1.0	0.696734670144	0.6967346699	2.44E − 10

Table 2: Performance of RKTM on problem 2:

x	Exact Solution	Computed Solution	Error
0.1	2.9401	2.940100001	-1E-09
0.2	2.7616	2.761600000	-1E-09
0.3	2.4681	2.468100001	-1E-09
0.4	2.0656	2.065599998	2E-09

5. CONCLUSIONS

Our computed results in both tables are very close to the exact solutions. This indicates the efficiency of the methods. The results are obtained at once which saves computer time and human effort. Also in this proposed method, the derivation is done only once and we can extend to higher orders unlike in conventional Backward Differentiation Formula.

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