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A Sixth Order Implicit Hybrid Backward Differentiation Formulae (HBDF) for Block Solution of Ordinary Differential Equations

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Abstract The Hybrid Backward Differentiation Formula (HBDF) for case K=5 was reformulated into continuous form using the idea of multistep collocation. Multistep Collocation is a continuous finite difference (CFD) approximation method by the idea of interpolation and collocation. The hybrid 5-step Backward Differentiation Formula (BDF) and additional methods of order $(6,6,6,6,6,)^T$ were obtained from the same continuous scheme and assembled into a block matrix equation which was applied to provide the solutions of IVPs over non-overlapping intervals.The continous form was immediately employed as block methods for direct solution of Ordinary Differential Equation ($y^{'} = f(x, y)$). Some benefits of this study are, the proposed block methods will be self starting and does not call for special predictor to estimate y' in the integrators and all the discrete methods obtained will be evaluated from a single continuous formula and its derivatives at various grids and off grid points. These study results help to speed up computation, also the requirement of a starting value and the overlap of solution model which are normally associated with conventional Linear Multistep Methods were eliminated by this approach. In conclusion, a convergence analysis of the derived hybrid schemes to establish their effectiveness and reliability was presented. Numerical example carried out on stiff problem further substantiates their performance.

Keywords Backward Differentiation Formula (BDF), Block Methods, Hybrid, Implicit, Multistep Collocation, Stiff

1. Introduction

Most real life problems that arise in various fields of study be it engineering or science are modelled as mathematical models before they are solved. These models often lead to differential equations.

A differential equation can simply be defined as an equation that contains a derivative. In other words, it's a relationship involving an independent variable x, a dependent variable y and one or more differential co-efficient of y with respect to x. An example of a differential equation is

$$y^{''} - xy = 0 \qquad (1)$$

Differential equations are of two types: An Ordinary Differential Equation (ODE) is one for which the unknown function (also known as dependent variable) is a function of a single independent variable. A Partial Differential Equation (PDE) is a differential equation in which the unknown function is a function of multiple independent variables and the equation involves its partial derivatives.

An ODE is classified according to the order of the highest derivative with respect to the dependent variable appearing in the equation. The most important cases for applications are the first and second order.

A numerical method is a difference equation involving a number of consecutive approximations y_{n+j}, $j = 0,1,2, \dots, k$ from which it will be possible to compute sequentially the sequence $\{y_n | n = 0,1,2 \dots, N\}$. Naturally this difference equation will also involve the function f. The integer k is called the step number of the method. For $k = 1$, it's called a 1- step method and for value of $k > 1$ it's called a multistep or k-step method.

If a computational method for determining the sequence $\{y_n\}$ takes the form of a linear relationship between $y_{n+j}, f_{n+j}, j = 0,1,2 \dots\dots\dots\dots k$ we call it a Linear Multistep Method of step number k or a Linear k-step method. These methods can be written in the general form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2)$$

where α_j, β_j are constants and we assume $\alpha_k \neq 0$ and that not both α_0 and β_0 are zero. Without loss of generality we let $\alpha_k = 1$. Explicit methods are characterized by $\beta_k = 0$ and implicit methods by $\beta_k = 1$. Explicit linear multistep methods are known as Adams-Bashforth methods, while implicit linear multistep methods are called Adams-moulton methods. These methods are generally called the Adams family.

Other famous classes of multistep methods aside the Adams family includes the predictor –corrector method and

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the Backward Differentiation Formula.

The Backward differentiation formula are implicit linear k-step method with regions of absolute stability large enough to make them relevant to the problem of stiffness.

Backward differentiation methods were introduced by Curtiss and Hirshfelder in 1952. For these methods $\beta_1 = \beta_2 = \cdot = \beta_k = 0$. These methods play a special role in the solution of stiff problems, despite not being A-stable for methods of order 3 or above. The most widely used adaptive codes for solving stiff differential equations are based on backward differentiation methods.

We consider the Initial Value Problem of the form

$$y' = f(x,y) \quad y(x_0) = y_0 \tag{3}$$

Where the solution y is assumed to be differentiable function on an interval $[x_0, b], b < \infty$. Many methods for solving (1) exists, one particular method is the Linear Multistep Method. Linear Multistep Methods require less evaluation of the derivative function f than one step methods in the range of integral $[x_0, b]$. For this reasons they have been very popular and important for solving (3) numerically. But these methods have certain limitations such as the overlap of solution models and the requirement of a starting value. Other limitations include they yield the discrete solution values $y_1, \dots, y_N$ hence uneconomical for producing dense output. A continuous formulation is desirable in this respect. The collocation method is probably the most important numerical procedure for the construction of continuous methods.

In this research paper, we derived the Block Hybrid Backward Differentiation Formulae (BHBDF) for ($k = 5$). The block methods were used to solve an Initial Value Problem directly without the need of a starting value. Their performances were compared with the analytical solution to the problem.

1.1. The Multistep Collocation (CMM) Method

Lambert (1973,1991) adopted the continous finite difference (CFD) approximation method by the idea of interpolation and collocation. Later, Lie and Norsett (1989), Onumanyi (1994,1999) referred to it as Multistep Collocation (MC). The method is presented below

$$\underline{a} = \left(a_{0,}a_1, \dots\dots, a_{(t+m-1)}\right)^T,$$

$$\varphi(x) = (\varphi_0(x), \varphi_1(x), \dots\dots, \varphi_{(t+m-1)})^T \tag{4}$$

where $a_{r,} r = 0, \dots\dots, t+m-1$ are undetermined constants, $\varphi_r(x\ \ )$ are specified basis functions, T denotes transpose of, t denotes the number of interpolation points and m denotes the number of distinct collocation points. We consider a continuous approximation (interpolant) Y(x) to y(x) in the form

$$y(x) = \sum_{r=0}^{t+m-1} a_r \varphi_r(x) = \underline{a}^T \varphi(x) \tag{5}$$

which is valid in the sub-intervals $x_n \le x \le x_{n+k}$, where n = 0,k……,N – k. The quantities

$$x_0 = a, x_N = b, k, m, n, t$$

$$and\ \varphi_r(x), r = 0,1 \dots., t+m-1$$

are specified values. The constant co-efficient a_r of (5) can be determined using the conditions

$$y(x_{n+j}) = y_{n+j,}\ j = 0,1, \dots., t-1 \tag{6}$$

$$y'(\overline{x}_j) = f_{n+j} \quad j = 0,1, \dots\dots, m-1 \tag{7}$$

Where

$$f_{n+j} = f(x_{n+j}, x_{n+j}) \tag{8}$$

The distinct collocation points $x_0, \dots\dots\dots\dots. x_{m-1}$, can be chosen freely from the set $[x_n, x_{n+k}]$. Equation (5), (6) and (7) are denoted by a single set of algebraic equations of the form

$$D\underline{a} = \underline{F} \tag{9}$$

Where

$$\underline{F} = (y_n, y_{n+1} \dots\dots. y_{n+t-1}, f_n, f_{n+1,} f_{n+m-1})^T \tag{10}$$

$$\underline{a} = D^{-1}\underline{F} \tag{11}$$

where $\underline{\underline{D}}$ is the non-singular matrix of dimension $(t+m)$

$$\underline{\underline{D}} = \begin{pmatrix} \varphi_0(x_n) & \cdots & \varphi_{t+m-1}(x_n) \\ \vdots & \vdots & \vdots \\ \varphi_0(x_{n+t-1}) & \cdots & \varphi_{t+m-1}(x_{n+t-1}) \\ & & \\ \varphi_0'(\overline{x}_0) & \cdots & \varphi_{t+m-1}'(\overline{x}_0) \\ \vdots & \vdots & \vdots \\ \varphi_0'(\overline{x}_{m-1}) & \cdots & \varphi_{t+m-1}'(\overline{x}_{m-1}) \end{pmatrix} \tag{12}$$

By substituting (11) into (5), we obtain the MC formula

$$y(x) = F^T \underline{\underline{C}}^T \varphi(x),\ \ x_n \le x \le x_{n+k}$$

$$n = 0, k, \dots\dots., N-k \qquad (13)$$

where

$$\underline{\underline{C}} \equiv \underline{\underline{D}}^{-1} = (c_{ij}), i,j = 1, \dots., t+m-1$$

$$\underline{\underline{C}} = \begin{pmatrix} c_{11} & \cdots & c_{1t} & c_{1,t+1} & \cdots & c_{1,t+m} \\ c_{21} & \cdots & c_{2t} & c_{2,t+1} & \cdots & c_{2,t+m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{t+m1} & \cdots & c_{t+m,t} & c_{t+m,t+1} & \cdots & c_{t+m,t+m} \end{pmatrix}$$

with the numerical elements denoted by $c_{ij\,,} i,j = 1, \dots., k+m$. By expanding $\underline{\underline{C}}^T\varphi(x)$ in (13) yields the following

$$y(x) = \sum_{j=0}^{t-1} \left(\sum_{r=0}^{t+m-1} C_{r+1,j+1}\varphi_r(x)\right)$$

$$+ \sum_{j=0}^{m-1} h\left(\sum_{r=0}^{k+m-1} \frac{C_{r+1,j+1}}{h}\varphi_r(x)\right) f_{n+j} \tag{14}$$

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f_{n+j} \tag{15}$$

a_r can be determined as follows:

$$y(x) = \left\{\sum_{r=0}^{t-1} \alpha_{j,r+1} y_{n+j} + h \sum_{j=0}^{m-1} \beta_{j,r+1} f_{n+j}\right\}\varphi_r(x)$$

2 Problem Formulation

For $K = 5$, the general form of the method upon addition of one off grid point is expressed as;

$$\bar{y}(x) = \alpha_1(x) y_n + \alpha_2(x) y_{n+1} + \alpha_3(x) y_{n+2}$$

$$+\alpha_4(x) y_{n+3} + \alpha_5(x) y_{n+4} + \alpha_6(x) y_{n+\frac{1}{2}} + h\beta_0(x) f_{n+5} \tag{16}$$

Recall from (9), $Da = F$

The matrix D of the proposed method is expressed as:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_n+h & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 & (x_n+h)^5 & (x_n+h)^6 \\ 1 & x_n+2h & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 & (x_n+2h)^6 \\ 1 & x_n+3h & (x_n+3h)^2 & (x_n+3h)^3 & (x_n+3h)^4 & (x_n+3h)^5 & (x_n+3h)^6 \\ 1 & x_n+4h & (x_n+4h)^2 & (x_n+4h)^3 & (x_n+4h)^4 & (x_n+4h)^5 & (x_n+4h)^6 \\ 1 & x_n+\frac{1}{2}h & (x_n+\frac{1}{2}h)^2 & (x_n+\frac{1}{2}h)^3 & (x_n+\frac{1}{2}h)^4 & (x_n+\frac{1}{2}h)^5 & (x_n+\frac{1}{2}h)^6 \\ 0 & 1 & 2x_n+10h & 3(x_n+5h)^2 & 4(x_n+5h)^3 & 5(x_n+5h)^4 & 6(x_n+5h)^5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+1/2} \\ f_{n+5} \end{bmatrix} \quad (17)$$

The matrix D in equation (17) which when solved by matrix inversion technique or Gaussian Elimination method will yield the continuous coefficients substituted in (16) to obtain continous form of the five step block hybrid BDF with one off step interpolation point.

$$\bar{y}(x) = \mathrm{A}y_n + \mathrm{B}y_{n+1} + \mathrm{C}y_{n+2} + \mathrm{D}y_{n+3} + \mathrm{E}y_{n+4} + \mathrm{F}y_{n+1/2} + G\mathrm{f}_{n+5} \quad (18)$$

Where

$$A = \begin{pmatrix} \frac{1}{10824}\frac{1}{h^6}\left(10824h^6 + 46190x_nh^5 + 166x_n^6 + 2645x_n^5h + 16111x_n^4h^2 + 47285x_n^3h^3 + 69019x_n^2h^4\right) \\ -\frac{1}{10824}\frac{1}{h^6}\left(46190h^5 + 996x_n^5 + 13225x_n^4h + 64444x_n^3h^2 + 141855x_n^2h^3 + 138038x_nh^4\right)x \\ +\frac{1}{10824}\frac{1}{h^6}\left(69019h^4 + 141855x_nh^3 + 96666x_n^2h^2 + 26450x_n^3h + 2490x_n^4\right)x^2 \\ -\frac{1}{10824}\frac{1}{h^6}\left(47285h^3 + 64444x_nh^2 + 26450x_n^2h + 3320x_n^3\right)x^3 \\ +\frac{1}{10824}\frac{1}{h^6}\left(16111h^2 + 13225x_nh + 2490x_n^2\right)x^4 - \frac{1}{10824}\frac{1}{h^6}\left(2645h + 996x_n\right)x^5 + \frac{83}{5412}\frac{x^6}{h^6} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{5412}\frac{1}{h^6}x_n\left(26520h^5 + 406x_n^5 + 6067x_n^4h + 33378x_n^3h^2 + 82427x_n^2h^3 + 86642x_nh^4\right) \\ -\frac{1}{5412}\frac{1}{h^6}\left(26520h^5 + 2436x_n^5 + 30335x_n^4h + 133512x_n^3h^2 + 247281x_n^2h^3 + 173284x_nh^4\right)x \\ +\frac{1}{5412}\frac{1}{h^6}\left(86642h^4 + 247281x_nh^3 + 200268x_n^2h^2 + 60670x_n^3h + 6090x_n^4\right)x^2 \\ -\frac{1}{5412}\frac{1}{h^6}\left(82427h^3 + 133512x_nh^2 + 60670x_n^2h + 8120x_n^3\right)x^3 \\ +\frac{1}{5412}\frac{1}{h^6}\left(33378h^2 + 30335x_nh + 6090x_n^2\right)x^4 - \frac{1}{5412}\frac{1}{h^6}\left(6067h + 2436x_n\right)x^5 + \frac{1}{h^6}\left(\frac{203}{2706}\right)x^6 \end{pmatrix}$$

$$\begin{aligned} C = (&-\frac{1}{16236}\frac{1}{h^6}x_n\left(25620h^5 + 782x_n^5 + 10917x_n^4h + 54281x_n^3h^2 + 115023x_n^2h^3 + 96497x_nh^4\right) \\ &+\frac{1}{16236}\frac{1}{h^6}\left(25620h^5 + 4692x_n^5 + 54585x_n^4h + 217184x_n^3h^2 + 345069x_n^2h^3 + 192994x_nh^4\right)x \\ &-\frac{1}{16236}\frac{1}{h^6}\left(96497h^4 + 345069x_nh^3 + 325686x_n^2h^2 + 109170x_n^3h + 11730x_n^4\right)x^2 \\ &+\frac{1}{16236}\frac{1}{h^6}\left(115023h^3 + 217124x_nh^2 + 109170x_n^2h + 15640x_n^3\right)x^3 \\ &-\frac{1}{16236}\frac{1}{h^6}\left(54281h^2 + 54585x_nh + 11730x_n^2\right)x^4 \\ &+\frac{1}{5412}\frac{1}{h^6}\left(3639h + 1564x_n\right)x^5 - \frac{1}{h^6}\left(\frac{391}{8118}\right)x^6) \end{aligned}$$

$$D = \begin{pmatrix} \frac{1}{27060}\frac{1}{h^6}x_n\left(15880h^5 + 722x_n^5 + 9385x_n^4h + 42410x_n^3h^2 + 80305x_n^2h^3 + 62438x_nh^4\right) \\ -\frac{1}{27060}\frac{1}{h^6}\left(15880h^5 + 4332x_n^5 + 46925x_n^4h + 169640x_n^3h^2 + 240915x_n^2h^3 + 124876x_nh^4\right)x \\ +\frac{1}{27060}\frac{1}{h^6}\left(240915x_nh^3 + 254460x_n^2h^2 + 93850x_n^3h + 10830x_n^4 + 62438h^4\right)x^2 \\ -\frac{1}{5412}\frac{1}{h^6}\left(16061h^3 + 33928x_nh^2 + 18770x_n^2h + 2888x_n^3\right)x^3 \\ +\frac{1}{5412}\frac{1}{h^6}\left(8482h^2 + 9385x_nh + 2166x_n^2\right)x^4 - \frac{1}{27060}\frac{1}{h^6}\left(9385h + 4332x_n\right)x^5 + \frac{1}{h^6}\left(\frac{361}{13530}\right)x^6 \end{pmatrix}$$

$$E = \begin{pmatrix} -\frac{1}{75768}\frac{1}{h^6}x_n\left(9210h^5 + 542x_n^5 + 6593x_n^4h + 27543x_n^3h^2 + 49213x_n^2h^3 + 36931x_nh^4\right) \\ +\frac{1}{75768}\frac{1}{h^6}\left(9210h^5 + 3252x_n^5 + 32965x_n^4h + 110172x_n^3h^2 + 147639x_n^2h^3 + 73862x_nh^4\right)x \\ -\frac{1}{75768}\frac{1}{h^6}\left(165258x_n^2h^2 + 65930x_n^3h + 8130x_n^4 + 147639x_nh^3 + 36931h^4\right)x^2 \\ +\frac{1}{75768}\frac{1}{h^6}\left(110172x_nh^2 + 65930x_n^2h + 10840x_n^3 + 49213h^3\right)x^3 \\ -\frac{1}{75768}\frac{1}{h^6}\left(27543h^2 + 32965x_nh + 8130x_n^2\right)x^4 \\ +\frac{1}{75768}\frac{1}{h^6}\left(6593h + 3252x_n\right)x^5 - \frac{1}{h^6}\left(\frac{271}{37884}\right)x^6 \end{pmatrix}$$

$$F = \begin{pmatrix} -\frac{64}{142065}\frac{1}{h^6}x_n\left(17880h^5 + 137x_n^5 + 2115x_n^4h + 12245x_n^3h^2 + 32925x_n^2h^3 + 40538x_nh^4\right) \\ +\frac{64}{142065}\frac{1}{h^6}\left(17880h^5 + 822x_n^5 + 10575x_n^4h + 48980x_n^3h^2 + 98775x_n^2h^3 + 81076x_nh^4\right)x \\ -\frac{64}{142065}\frac{1}{h^6}\left(2055x_n^4 + 21150x_nh^3 + 73470x_n^2h^2 + 98775x_nh^3 + 40538h^4\right)x^2 \\ +\frac{64}{28413}\frac{1}{h^6}\left(548x_n^3 + 4230x_n^2h + 9796x_nh^2 + 6585h^3\right)x^3 \\ -\frac{64}{28413}\frac{1}{h^6}\left(411x_n^2 + 2115x_nh + 2449h^2\right)x^4 \\ +\frac{64}{47355}\frac{1}{h^6}\left(274x_n + 705h\right)x^5 - \frac{1}{h^6}\left(\frac{8768}{142065}\right)x^6 \end{pmatrix}$$

$$G = \begin{pmatrix} \frac{1}{2706}\frac{1}{h^5}x_n\left(24h^5 + 2x_n^5 + 21x_n^4h + 80x_n^3h^2 + 135x_n^2h^3 + 98x_nh^4\right) \\ -\frac{1}{2706}\frac{1}{h^5}\left(24h^5 + 12x_n^5 + 105x_n^4h + 320x_n^3h^2 + 405x_n^2h^3 + 196x_nh^4\right)x \\ +\frac{1}{2706}\frac{1}{h^5}\left(30x_n^4 + 210x_nh^3 + 480x_n^2h^2 + 405x_nh^3 + 98h^4\right)x^2 \\ -\frac{5}{2706}\frac{1}{h^5}\left(8x_n^3 + 42x_n^2h + 64x_nh^2 + 27h^3\right)x^3 \\ +\frac{5}{2706}\frac{1}{h^5}\left(6x_n^2 + 21x_nh + 16h^2\right)x^4 \\ -\frac{1}{902}\frac{1}{h^5}\left(4x_n + 7h\right)x^5 + \frac{1}{h^5}\left(\frac{1}{1355}\right)x^6 \end{pmatrix}$$

Evaluating (18) at point $x = x_{n+5}$ and its derivative at $x = x_{n+4}, x = x_{n+3}, x = x_{n+2}, x = x_{n+1}, x = x_{n+1/2}$ yields the following six discrete hybrid schemes which are used as block integrator:

$$\frac{2025}{451}y_{n+1} - \frac{1800}{451}y_{n+2} + \frac{1620}{451}y_{n+3} - \frac{8100}{3157}y_{n+4} + y_{n+5} - \frac{10240}{3157}y_{n+\frac{1}{2}} = -\frac{324}{451}y_n + \frac{180}{451}hf_{n+5}$$

$$-y_{n+1} - \frac{30520}{44590}y_{n+2} + \frac{8526}{44590}y_{n+3} - \frac{1580}{44590}y_{n+4} + \frac{77824}{44590}y_{n+\frac{1}{2}} = \frac{9660}{44590}y_n - \frac{47355}{44590}hf_{n+1} - \frac{105}{44590}hf_{n+5}$$

$$\frac{293580}{71540}y_{n+1} - y_{n+2} - \frac{106092}{71540}y_{n+3} + \frac{14895}{71540}y_{n+4} - \frac{160768}{71540}y_{n+\frac{1}{2}} = -\frac{29925}{71540}y_n - \frac{189420}{71540}hf_{n+1} + \frac{840}{71540}hf_{n+5}$$

$$-\frac{130200}{78904}y_{n+1} + \frac{168350}{78904}y_{n+2} - y_{n+3} - \frac{27075}{78904}y_{n+4} + \frac{85504}{78904}y_{n+\frac{1}{2}} = \frac{17675}{78904}y_n - \frac{94710}{78904}hf_{n+3} - \frac{1050}{78904}hf_{n+5}$$

$$\frac{574280}{334925}y_{n+1} - \frac{559580}{334925}y_{n+2} + \frac{636216}{334925}y_{n+3} - y_{n+4} - \frac{403456}{334925}y_{n+\frac{1}{2}} = -\frac{87465}{334925}y_n - \frac{189420}{334925}hf_{n+4} + \frac{11760}{334925}hf_{n+5}$$

$$-\frac{4917150}{3141632}y_{n+1} + \frac{1055950}{3141632}y_{n+2} - \frac{353682}{3141632}y_{n+3} + \frac{69975}{3141632}y_{n+4} + y_{n+\frac{1}{2}} = -\frac{1003275}{3141632}y_n - \frac{2020480}{3141632}hf_{n+1/2} + \frac{4900}{3141632}hf_{n+5} \quad (19)$$

Equation (19) constitute the members of a zero-stable block integrators of order $(6,6,6,6,6,6)^T$ with

$$C_7 = \left[-\frac{135}{3157}124, -\frac{1533}{2}, \frac{2705}{4}, -4417, -\frac{51555}{8}\right]^T$$

as the error constants respectively. To start the integration process with n=0, we use (19) and this produces $y_1, y_{1/2}, y_2, y_3, y_4$, and y_5 simultaneously without the need of any starting method (predictor).

2.1. Stability Analysis

Following Fatunla (1992; 1994),that defined the block method to be zero-stable provided the roots $R_{ij} = 1(1)k$ of the first characteristic polynomial $\rho(R)$ specified as

$$\rho(R) = det\left|\sum_{i=0}^{k} A^{(i)} R^{k-i}\right| = 0 \quad (20)$$

satisfies $|R_j| \leq 1$,the multiplicity must not exceed 2.

The block methods proposed in equation (19) for $k = 5$ are put in the matrix equation form and for easy analysis the result was normalized to obtain

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (21)$$

The first characteristic polynomial of the block method is given by $\rho(R) = \det(RA^0 - A^1)$ Substituting the A^0 and A^1 into the function above gives

$$\rho(R) =$$

$$\det\left[R\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right]$$

$$= det\left[\begin{bmatrix} R & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right]$$

$$= R^4(R(R-1)) - 0 = 0 \quad (22)$$

$$\Rightarrow R_1 = R_2 = R_3 = R_4 = R_5 = 0 \text{ or } R_6 = 1$$

From equation (22) the hybrid method is zero stable and consistent since the order of the method $p = 6 > 1$. And by Henrici (1962); the hybrid method is convergent.

3. Problem Solution

To illustrate the performance of our proposed methods we will compare their performance with analytical results. Consider the initial value problem

$$y' = \lambda(y - x) + 1, y(0) = 1$$

The problem is stiff in nature for negative λ values and it has analytical solution $y(x) = e^{\lambda x} + x$.

The problem is solved with $\lambda = -5, and\ \lambda = -20$ and steplength $h = 0.01$ using the Block Hybrid Backward Differentiation Formulae (BHBDF) for $k = 5$. The results were compared with analytical method and Block Hybrid Backward Differentiation Formulae (BHBDF) for $k = 4$.

Table 1. Proposed (BHBDF) for K= 5, λ = -5

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.961229424	0.961229424	0
2	0.02	0.924837418	0.924837418	0
3	0.03	0.890707976	0.890707976	0
4	0.04	0.858730753	0.858730752	1E-9
5	0.05	0.828800783	0.828800783	0
6	0.06	0.80081822	0.80081822	0
7	0.07	0.774688089	0.774688089	0
8	0.08	0.750320046	0.750320046	0
9	0.09	0.727628151	0.727628151	0
10	0.1	0.706530659	0.706530659	0

Table 2. Proposed (BHBDF) for K=5, λ= -20

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.828730753	0.828730837	8.4E-8
2	0.02	0.690320046	0.690320135	8.9E-8
3	0.03	0.578811636	0.578811692	5.6E-8
4	0.04	0.489328964	0.489329036	7.2E-8
5	0.05	0.417879441	0.41787941	3.1E-8
6	0.06	0.361194211	0.361194218	7.0E-9
7	0.07	0.316596963	0.316596976	1.3E-8
8	0.08	0.281896518	0..281896521	3.0E-9
9	0.09	0.255298888	0. 255298901	1.3E-8
10	0.1	0.235335283	0.23533526	2.3E-8

Table 3. Proposed (BHBDF) for K=4, λ=-5

X	Exact Value	Approximate value	Error
0.00	1.000000000	1.000000000	0
0.01	0.961229424	0.961229424	5.03E-10
0.02	0.924837418	0.924837418	3.90E-11
0.03	0.890707976	0.890707976	4.27E-10
0.04	0.858730753	0.858730753	8.0E-11
0.05	0.828800783	0.828800783	7.40E-11
0.06	0.80081822	0.800818221	3.17E-10
0.07	0.774688089	0.77468809	2.8E-10
0.08	0.750320046	0.750320046	3.7E-11
0.09	0.727628151	0.727628152	3.77E-10
0.1	0.706530659	0.706530667	2.86E-10

Table 4. Proposed (BHBDF) for K=4, λ= -20

X	Exact Value	Approximate value	Error
0.00	1.000000000	1.000000000	0
0.01	0.828730753	0.828769259	3.8506896E-5
0.02	0.690320046	0.690432454	1.12408945E-4
0.03	0.578811636	0.578993079	1.81443769E-4
0.04	0.489328964	0.489416887	8.7923593E-5
0.05	0.417879441	0.417989621	1.10180194E-4
0.06	0.361194211	0.361365196	1.70985282E-4
0.07	0.316596963	0.316826495	2.29531969E-4
0.08	0.281896518	0..282023466	1.26948875E-4
0.09	0.255298888	0. 255440879	1.4199104E-4
0.1	0.235335283	0.235532264	1.96981522E-4

In this research work, the continuous formulation of linear multistep methods through matrix inversion approach of (Onumanyi et,al 1994); Yahaya and Adegboye (2007); Sokoto (2009); Yahaya etal (2010); Yahaya and Umar (2010) was carried out .This was extended to the construction of a family of block hybrid backward differentiation formula (BHBDF) for step number $k = 5$ with one off grid point at $x = x_{n+\mu}, \mu = 1/2$, which is suitable for stiff problems. Convergence Analysis of the resulting discrete block hybrid method was done using the zero stability theory of fatunla (1992; 1994) for k step block methods. Numerical Experiment for stiff initial value problem was carried out. Results obtained for the problem implemented by our present method was tabulated. The BHBDF for K= 5 is of higher accuracy and performance than BHBDF for K= 4 for both Eigen values (λ = -5 and λ = -20). The block methods produce accurate results when compared with analytical results.

4. Conclusions

We have derived the hybrid form of the Backward Differentiation Formulae (BDF) for $k = 5$. The idea of Multistep Collocation (MC) was used to reformulate the derived hybrid formulae into continous form which were immediately employed as block methods for direct solution of $y' = f(x, y)$. A convergence analysis of the discrete hybrid methods to establish their effectiveness and reliability is presented. The methods were tested on stiff IVP and shown to perform satisfactorily without the requirement of any starting method. The newly constructed methods speed up computation and eliminate the overlap of solution model.

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