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# An Evaluation of Lotka-Volterra Predator-Prey Model

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**Abstract**
Many interesting dynamics of biological systems, more often than not, involve interactions between species. One of such interactions exists between predators and prey; and the result of this interaction plays a significant role in the dynamics of their respective populations. One of the earliest and simplest mathematical models to describe this interaction is the Lotka-Volterra model. In this paper, we review the derivation, evaluation and analysis of the Lotka-Volterra model, with a view to bringing to fore its realism, effectiveness and possible shortcomings; as well as suggesting areas of further improvement in the model.
**Keywords:** Equilibrium state, Carrying capacity, Predator, Prey

## Introduction

Predation typifies an antagonistic ecological interaction in which one specie, the predator, takes advantage of another specie, the prey. The prey serves as a source of food for the predator. This interaction is advantageous on the predator's population, as predators consume more prey, they are able to produce more predators. One of the first models to describe this interaction is the Lotka-Volterra equations, otherwise known as the predator-prey equations. It was developed independently by Alfred J Lotka in 1925 and Vito Volterra in 1926. Lotka (1925) described a hypothetical chemical reaction in which the chemical concentrations oscillate. Volterra (1926) proposed a differential equation model to explain the observed increase in predator fish (and corresponding decrease in prey fish) in the Adriatic Sea during World War I. This model, is a system of two differential equations, one describing the prey's population, and the other describing the predator's population. It describes predator-prey dynamics in their simplest case (one predator population, one prey population).

The Lotka-Volterra model, considered as the basis for more complex predator-prey models, has attracted the attention of many researchers who have been making significant progress toward improving the model. Holling (1959), developed a model (often called 'disc equation') of functional response which remains very popular among ecologists. This model highlights the principal of time budget in behavioural ecology. Rogers (1972) applies the model of Holling, originally developed for predator-prey systems, to host-parasite systems. Its assumption was based on two kinds of limitations in host-parasitoid interactions, i.e., limited parasitoid fecundity and limited search rate.

## Methodology

We analyze the procedure leading to derivation of the Lotka-Voterra model, beginning with the following assumptions.

- The prey population will grow exponentially when the predator is absent.
- The predator population will starve in the absence of prey population (as opposed to switching to another type of prey).

- Predators can consume infinite quantities of prey.
- The prey has unlimited food supply.
- The predator's growth rate depends on the prey it catches.

From the assumptions above, we can deduce the following parameters:
$a$ = growth rate of prey
$b$ = attack rate
$c$ = predator mortality rate
$d$ = predator efficiency at turning food into offspring
$t$ = time
and variables:
$x$ = number of prey
$y$ = number of predators
With these parameters defined, the model will be written in terms of two autonomous differential equations, each describing the dynamics of the predator and prey populations respectively. In the absence of prey, the predators are left without food resources since it is assumed they have no any other source of food. Ultimately, their population is expected to decrease exponentially, as described by the following differential equation:

$$\frac{dy}{dt} = -cy \qquad (4)$$

Equation (4) above employs the product of the number of predators ($y$), the predator mortality rate ($c$) and the negative sign to describe the exponential decay of the predator population ($y$) with respect to time ($t$). In the presence of prey, the population of predators is expected to increase due to their consumption of prey. This increase in dependent upon the predator birth rate ($d$) multiplied by the product of the number of prey ($x$) and the number of predators ($y$), as represented by the following differential equation:

$$\frac{dy}{dt} = dxy \qquad (5)$$

Equations (4) and (5) are combined to obtain the following equation:

$$\frac{dy}{dt} = dxy - cy = -y(c - dx) \qquad (6)$$

which effectively describes the predator population dynamics.
We develop a model to describe the dynamics of the prey population by first considering what happens to the when there are no predators. In the absence of predators, the prey population is expected to increase exponentially, since it is assumed that the prey has unlimited food supply. This exponential growth is described by the following differential equation:

$$\frac{dx}{dt} = ax \qquad (7)$$

In a biological ecosystem, however, predators and prey are bound to live together. This living often results in interactions between the two species, which also results in the activity of predation, and which ultimately plays a vital role in determining the populations of these species. The impact of this interaction is determined by the predator attack rate ($b$) multiplied by the number of prey ($x$) and the number of predators ($y$). This is described by the equation below:

$$\frac{dx}{dt} = -bxy \qquad (8)$$

Combining equations (7) and (8) produces the prey population dynamics:

$$\frac{dx}{dt} = ax - bxy = x(a - by) \qquad (9)$$

Equations (6) and (9) together make up the Lotka-Volterra predator-prey model, which describes predator and prey population dynamics in the presence of one another. We can interpret the dynamics of the system thus; an increase in the number of predators ($y$) implies more encounters with the prey, and consequently more predation is expected to take place. This increase in attack rate, however,

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has an adverse effect (a decrease) in the number of prey ($x$), which also causes $y$ (and therefore $bxy$) to decrease. While $bxy$ is decreasing, the number of prey appreciates, and $x$ increases. Consequently, the predator population appreciates as well. These dynamics continue in a cycle of growth and decline.

## POPULATION EQUILIBRIUM

Equilibrium is the condition of mutual co-existence in the sense that neither of the population levels is changing. It means that the mortality and birth rates of both the predator and prey are equal (individually) over a period of time. The equilibrium state is attained when both differential equations vanish, i.e.,

$$\frac{dy}{dt} = -y(c - dx) = 0$$

$$\frac{dx}{dt} = x(a - by) = 0$$

Solving the above system for $x$ and $y$ yields the following results

$$(y = 0,\ x = 0) \qquad (10)$$

$$\left(y = \frac{a}{b},\ x = \frac{c}{d}\right) \qquad (11)$$

Thus, the two solutions (10) and (11) describe two points of equilibria for the system. The first equilibrium, equation (10), occurs when both populations are at 0. This can only be interpreted as a point of extinction, as the populations remain at 0 indefinitely. The second equilibrium, equation (11), can be interpreted to be a fixed point at which both the predator and prey population are sustained indefinitely.

**Model Evaluation**

A very important aspect of any model design is evaluation, to determine whether the model produces realistic results as predicted. A solution of the Lotka-Volterra model implies solving the two differential equations represented by equations (6) and (9). We perform four separate experiments on the model, by providing numerical solutions using Euler's method, described as follows

$$y_{n+1} = y_n + hf_n = y_n + h\{y_n(dx_n - c)\} \qquad (12)$$

and

$$x_{n+1} = x_n + hf_n = x_n + h\{x_n(a - by_n\} \qquad (13)$$

for the predator and prey equations respectively.

<u>Experiment 1</u>

The following values are substituted into equations (12) and (13)

$x_0 = 70$

$y_0 = 30$

$h = 0.05$

$a = 0.5$

$b = 0.04$

$c = 0.4$

$d = 0.02$

A graph of results obtained from experiment 1is presented in Figure 1

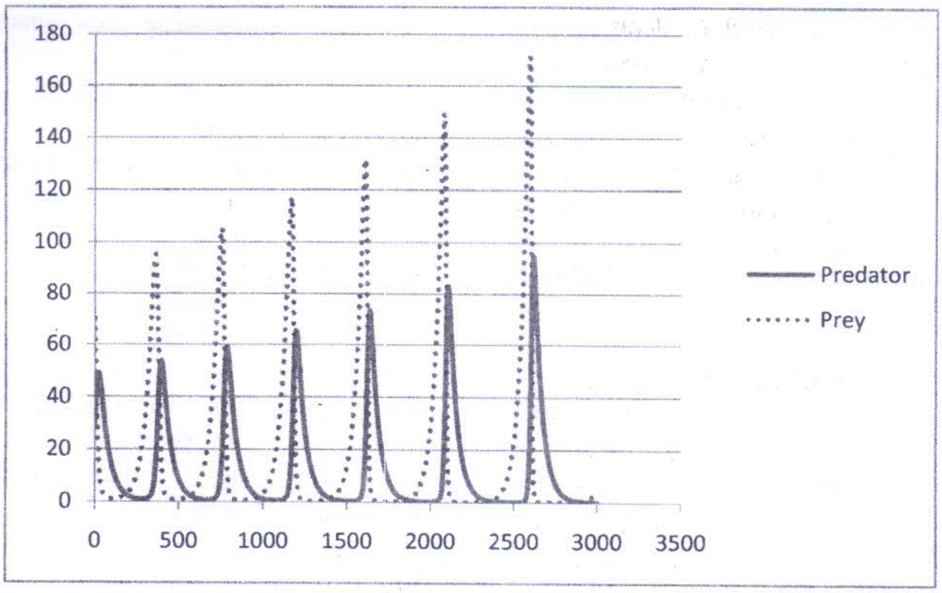


Figure 1 Cyclical relationship for predator and prey populations

Experiment 2

The following values are assigned for the parameters and in equation (12) and (13)

$x_0 = 40$

$y_0 = 0$

$h = .0.05$

$a = 0.02$

$b = 0$

$c = 0$

$d = 0$

where the results are summarized in the graph of Figure 2.

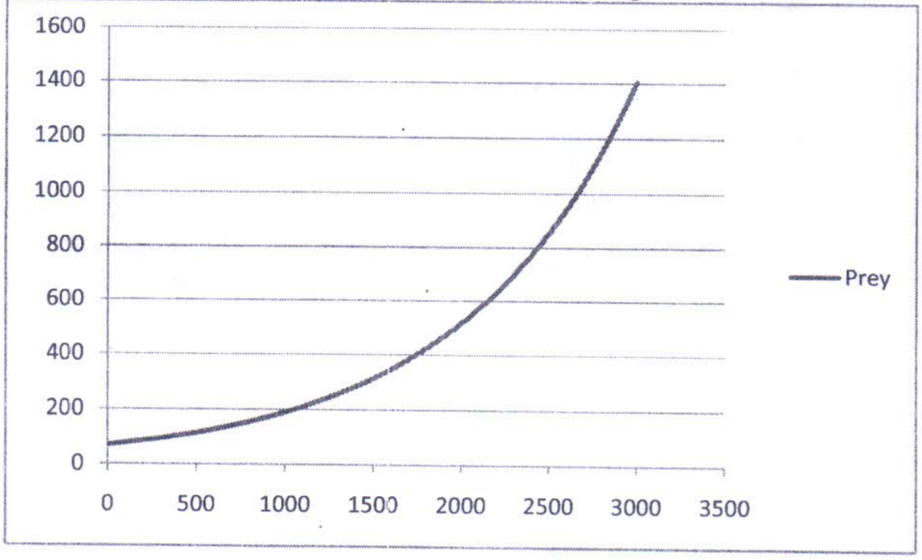


Figure 2 Exponential growth of prey when there are no predators

Experiment 3

The following values are assigned for the parameters and variables.

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$x_0 = 0$
$y_0 = 30$
$h = 0.05$
$a = 0$
$b = 0$
$c = 0.1$
$d = 0$
with its corresponding graph as shown in Figure 3 below.

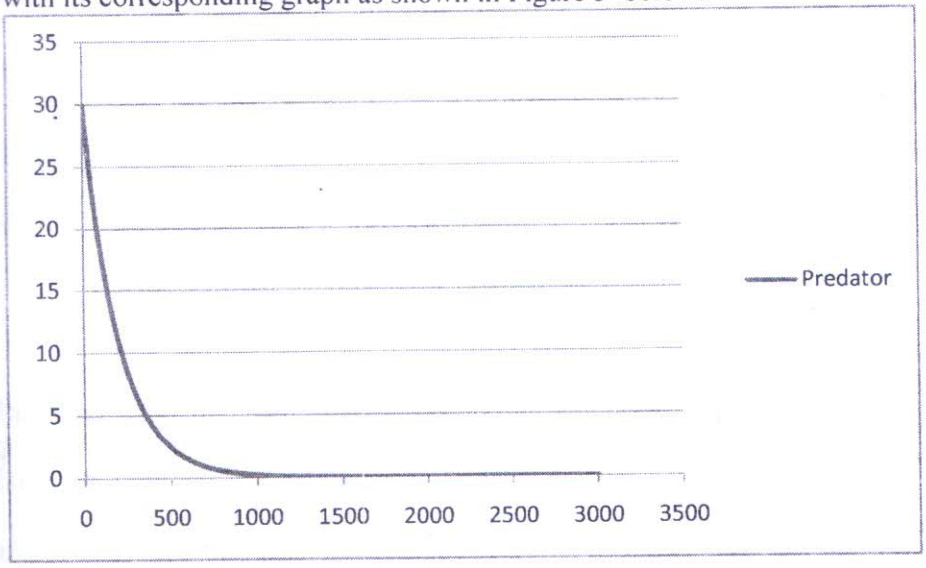


Figure 3 Exponential decay of predators when there are no prey

Experiment 4

The following values are assigned for the parameters and variables.
$x_0 = 40$
$y_0 = 20$
$h = 0.05$
$a = 0.03$
$b = 0.2$
$c = 0.4$
$d = 0.01$

Figure 4 below depicts the graph of what happens after performing experiment 4.

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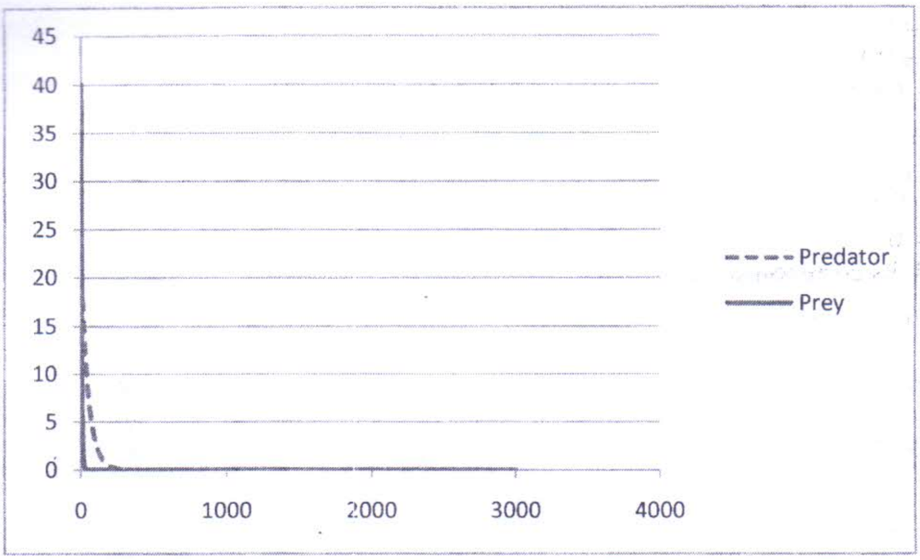


Figure 4 Population equilibrium

**Discussion of results**

In all the graphs of figures 1 – 4 the $y$ – axis represents the populations of predators and prey, while the $x$ – axis represents time. The graph of figure 1 depicts the cyclical relationship predicted by the model for predator and prey populations, with the graph of the predator lagging behind that of prey.

Figure 2 shows the graph of what happens when there is no predator in the ecosystem. As predicted, there is an exponential increase in the growth of the prey, and this increase has no bound unless an external factor is introduced into the system. Figure 3 is an illustration of the exponential decay of the predator population in the absence of prey. And lastly, figure 4 shows the graph of both predators and prey tending toward 0, never to recover again. It depicts the situation of extinction for the two species, as predicted by the unstable equilibrium state of the model.

**Conclusion**

The Lotka-Volterra model is the simplest predator–prey model based on sound mathematical principles. It provides an intriguing example for later research works in the analysis of predator-prey dynamics. Yet, in its original form it is fraught with significant problems. One of the main problems with this model rears its head when there are no predators in the system. In that situation, the prey population grows exponentially without bound, which is contrary to realistic situations. To remedy this situation, it is suggested that a carrying capacity term be incorporated into the model, to allow for a maximum prey population size that a particular environment can support. Furthermore, the two equilibrium points are unstable, necessitating the seemingly endless cycle of the predator and prey populations without settling down quickly. Another problem with this model is some of its unrealistic assumptions. For example, the prey populations are assumed to have unlimited food supply, contrary to realism; and predators are assumed to consume infinite quantities of prey.

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**References**

Holling, C S (1959), *Some Characteristics of Simple Types of Predation and Parasitism,* Canad. Entomol., 91: 385-398

Lotka, A J (1925), *Elements of Physical Biology*, Baltimore, William & Wilkins Co.

Rogers, D J (1972), *Random Search and Insect Population Models,* J. Animal Ecol., 41: 369-383

Volterra, V, (1926), *Variazioni e fluttuazioni del numero d'individui in specie animali sconviventi,* Mem. R. Accad. Naz. Dei Lincei. Ser. VI, vol.2