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Improving Accuracy Through the Three Steps Block Methods For Direct Solution of Second Order Initial Value Problem Using Interpolation and Collocation Approach

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Abstract

This paper presents three-step block method for direct solution of second order initial value problems of ordinary differential equations. The collocation and interpolation approach was adopted to generate a continuous block method using power series as basis function. The properties of the proposed approach such as order, error constant, zero-stability, consistency and convergence were also investigated. The proposed method competes favorably with exact solution and the existing methods.

Keywords: collocation, interpolation, power series, zero-stability, convergence, block method

1. Introduction

Differential equation is an equation involving a relationship between an unknown function and one or more of its derivatives. Many numerical methods have been constructed in response to the need to find numerical approximations to ordinary differential equation arising from "real-life" problems in applied mathematics and it became inevitable tools used in solving ordinary differential equations. Since, the advent of computers, the numerical solution of Initial Value Problems (IVP) for Ordinary Differential Equations (ODEs) has been the subject of research by numerical analysts, especially methods for the numerical solutions of the Special Second Order Ordinary Differential Equations of the form.

$$y'' = f(x,y), \quad y(x_0) = y_0, \qquad y'(x_0) = y'_0 \qquad (1)$$

In which the first derivatives y' is absent. In general, the special second order ordinary differential equation of (1) can be solved directly without the first derivative being explicitly present.
However, only a limited number of numerical methods are available for solving (1) directly without reducing to a first order system of ordinary differential equations. This study, therefore

propose a block multistep method of step-length $k = 3$ for the direct solution of second order initial value problems of ordinary differential equations, which is self-starting, eliminating the use of predictors which increase the accuracy of the method in terms of error.
An extensive discussion can be seen in (Awoyemi, 1999; Jator, 2007, Aladeselu, 2007)
It was noticed that the reduction process has a lot of setbacks such as difficulties in writing computer program for the method, computational burden which affects the accuracy of the method in terms of error and depletion of human effort.

Therefore, it will be appropriate and more efficient if direct method of solving (1) is employed as suggested in (Onumanyi et al., 2001; Adesanya et al., 2009; Awoyemi et al., 2011) and Adeniran et al., 2015). Much and considerable attention have been dedicated to solving higher order ordinary differential equations directly without being reduced to system of first order ordinary differential equation.

2. Problem Formulation

We consider the power series of the form

$$y(x) = \sum_{j=0}^{k+2} a_j \ x^j \qquad (2)$$

as approximate solution to the general second order problems

$$y''(x) = f(x, y, y') \colon y(x_0) = y_0 , \qquad (3)$$

Where the Step-number k in equation (2) is 3, the first and second derivatives of (2) are

$$y'(x) = \sum_{j=1}^{k+2} j \ a_j \ x^{j-1} \qquad (4)$$

$$y''(x) = \sum_{j=2}^{k+2} j \ (j-1) a_j \ x^{j-2} \qquad (5)$$

If m denotes the number of interpolation points and n denotes the number of distinct collocation points. Then equation (2) is interpolated at the point $x = x_n$, i = 0(1) 1($i{:}\ e\ m = 2$), while equation (3) is collocated at the point $x = x_{n+1}$, i =0 (1),3 ($i{:}\ e\ \ n = 4$)
These interpolation and collocation points will determine the degree of polynomial (p) which is given as

$$p = m + n - 1 \qquad (6)$$

The general form of Power series is given by

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4\ x^4 + a_5 x^5 + \cdots \dots \dots + a_n x^n \qquad (7)$$

Truncating after the $5th$ term since from equation (6) $p = 2 + 4 - 1 = 5$, yields

$$y(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4\ x_n^4 + a_5\ x_n^5 \qquad (8)$$

At $x = x_{n+1}$ equation (8) now becomes

$$y(x_{n+1}) = a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4\ x_{n+1}^4 + a_5\ x_{n+1}^5 \qquad (9)$$

Taking the second derivative of equation (8) and applying all the collocation points i =0 (1),3 gives rise to equation (10)

$$\left.\begin{aligned} y''(x_n) &= 2a_2 + 6a_3x_n + 12a_4x_n^2 + 20a_5x_n^3 \\ y''(x_{n+1}) &= 2a_2 + 6a_3x_{n+1} + 12a_4x_{n+1}^2 + 20a_5x_{n+1}^3 \\ y''(x_{n+2}) &= 2a_2 + 6a_3x_{n+2} + 12a_4x_{n+2}^2 + 20a_5x_{n+2}^3 \\ y''(x_{n+3}) &= 2a_2 + 6a_3x_{n+3} + 12a_4x_{n+3}^2 + 20a_5x_{n+3}^3 \end{aligned}\right\} \quad (10)$$

Equation (8), (9) and (10) in matrix form will help to determine the inverse of the matrix which will generate the continuous coefficients for the propose method

$$\mathrm{A} = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 \end{bmatrix} \qquad (11)$$

Finding the inverse of the matrix A using maple software package gives rise to the matrix C

$$C = \begin{bmatrix} \frac{x_n+h}{h} & -\frac{x_n}{h} & \frac{3x_n^5+30hx_n^4+110h^2x_n^3+180h^3x_n^2+97h^4x_n}{360h^3} & \frac{-3x_n^5-25hx_n^4-60h^2x_n^3+38h^4x_n}{120h^3} & \frac{3x_n^5+20hx_n^4+30h^2x_n^3-13h^4x_n}{120h^3} & \frac{-3x_n^5-15hx_n^4-20h^2x_n^3+8h^4x_n}{360h^3} \\ -\frac{1}{h} & \frac{1}{h} & -\frac{3x_n^4+120hx_n^3+330h^2x_n^2+360h^3x_n+97h^4}{360h^3} & \frac{15x_n^4+100hx_n^3+180h^2x_n^2-38h^4}{120h^3} & \frac{-15x_n^4-80hx_n^3-90h^2x_n^2+13h^4}{120h^3} & \frac{15x_n^4+60hx_n^3+60h^2x_n^2-8h^4}{360h^3} \\ 0 & 0 & \frac{x_n^3+6hx_n^2+11h^2x_n+6h^3}{12h^3} & -\frac{x_n^3+5hx_n^2+6h^2x_n}{4h^3} & \frac{x_n^3+4hx_n^2+3h^2x_n}{4h^3} & -\frac{x_n^3+3hx_n^2+2h^2x_n}{12h^3} \\ 0 & 0 & -\frac{3x_n^2+12hx_n+11h^2}{36h^3} & \frac{3x_n^2+10hx_n+6h^2}{12h^3} & -\frac{3x_n^2+8hx_n+3h^2}{12h^3} & \frac{3x_n^2+6hx_n+2h^2}{36h^3} \\ 0 & 0 & \frac{x_n+2h}{24h^3} & -\frac{3x_n+5h}{24h^3} & \frac{3x_n+4h}{24h^3} & -\frac{x_n+h}{24h^3} \\ 0 & 0 & -\frac{1}{120h^3} & \frac{1}{40h^3} & -\frac{1}{40h^3} & \frac{1}{120h^3} \end{bmatrix}$$

(12)

The general form of the propose second order method is given as (13)

$$y(x) = \sum_{i=0}^{1} \alpha_i\,(x)\,y_{n+i} + h^2 \sum_{i=0}^{3} \beta_i\,(x)\,f_{n+i} \qquad (13)$$

Recall that ($m = 0{,}1\ for\ y_{n+i}$) and ($n = 0{,}1{,}2{,}3\ for\ f_{n+i}$) Therefore substituting these values in (13) we have (14)

$$y(x) = \alpha_0\,(x)y_n + \alpha_1\,(x)y_{n+1} + h^2\{\beta_0\,(x)f_n + \beta_1\,(x)f_{n+1} + \beta_2\,(x)f_{n+2} + \beta_3\,(x)f_{n+3} \quad (14)$$

The elements of C ($\alpha_i\ and\ \beta_i$) are used to generate the coefficients

$$\alpha_0\,(x) = c_{11}x^0 + c_{21}x^1 + c_{31}x^2 + c_{41}x^3 + c_{51}x^4 + c_{61}x^5$$

$$= \frac{x_{n+h}}{h}x^0 + (-\frac{1}{h})x^1 + 0.x^2 + 0.x^3 + 0.x^4 + 0.x^5 = \frac{x_{n+h}}{h} - \frac{x}{h} \quad (15)$$

$$\alpha_1\,(x) = c_{12}x^0 + c_{22}x^1 + c_{32}x^2 + c_{42}x^3 + c_{52}x^4 + c_{62}x^5$$

$$= -\left(\frac{x_{n+1-h}}{h}\right)x^0 + \left(\frac{1}{h}\right)x + 0.x^2 + 0.x^3 + 0.x^4 + 0.x^5 = \frac{-x_n}{h} + \frac{x}{h} \quad (16)$$

$$\beta_0\,(x) = c_{13}x^0 + c_{23}x^1 + c_{33}x^2 + c_{43}x^3 + c_{53}x^4 + c_{63}x^5$$

$$= \left(\frac{3x_n^5+30hx_n^4+110h^2x_n^3+180h^3x_n^2+97h^4x_n}{360h^3}\right)x^0 - \left(\frac{3x_n^4+120hx_n^3+330h^2x_n^2+360h^3x_n+97h^4}{360h^3}\right)x + \left(\frac{x_n^3+6hx_n^2+11h^2x_n+6h^3}{12h^3}\right)x^2 -$$

$$\left(\frac{3x_n^2+12hx_n+11h^2}{36h^3}\right)x^3 + \left(\frac{x_n+2h}{24h^3}\right)x^4 - \frac{1}{120h^3}x^5$$

$$\beta_0\,(x) = -\left(\frac{3(x-x_n)^5-30h(x-x_n)^4+110h^2(x-x_n)^3-180h^3(x-x_n)^2+97h^4(x-x_n)}{360h^3}\right) \quad (17)$$

$$\beta_1\,(x) = c_{14}x^0 + c_{24}x^1 + c_{34}x^2 + c_{44}x^3 + c_{54}x^4 + c_{64}x^5$$

$$= \left(\frac{-3x_n^5-25hx_n^4-60h^2x_n^3+38h^4x_n}{120h^3}\right)x^0 + \left(\frac{15x_n^4+100hx_n^3+180h^2x_n^2-38h^4}{120h^3}\right)x - \left(\frac{x_n^3+5hx_n^2+6h^2x_n}{4h^3}\right)x^2 + \left(\frac{3x_n^2+10hx_n+6h^2}{12h^3}\right)x^3 -$$

$$\left(\frac{3x_n+5h}{24h^3}\right)x^4 + \left(\frac{1}{40h^3}\right)x^5$$

$$\beta_1\,(x) = -\left(\frac{-3(x-x_n)^5+25h(x-x_n)^4-60h^2(x-x_n)^3+38h^4(x-x_n)}{120h^3}\right) \quad (18)$$

$$\beta_2\,(x) = c_{15}x^0 + c_{25}x^1 + c_{35}x^2 + c_{45}x^3 + c_{55}x^4 + c_{65}x^5$$

$$= \left(\frac{3x_n^5+20hx_n^4+30h^2x_n^3-13h^4x_n}{120h^3}\right)x^0 + \left(\frac{-15x_n^4-80hx_n^3-90h^2x_n^2+13h^4}{120h^3}\right)x + \left(\frac{x_n^3+4hx_n^2+3h^2x_n}{4h^3}\right)x^2 - \left(\frac{3x_n^2+8hx_n+3h^2}{12h^3}\right)x^3 +$$

$$\left(\frac{3x_n+4h}{24h^3}\right)x^4 - \left(\frac{1}{40h^3}\right)x^5$$

$$\beta_2\,(x) = \left(\frac{-3(x-x_n)^5+20h(x-x_n)^4-30h^2(x-x_n)^3+13h^4(x-x_n)}{120h^3}\right) \quad (19)$$

$$\beta_3(x) = c_{16}x^0 + c_{26}x^1 + c_{36}x^2 + c_{46}x^3 + c_{56}x^4 + c_{66}x^5$$

$$= \left(\frac{-3x_n^5-15hx_n^4-20h^2x_n^3+8h^4x_n}{360h^3}\right)x^0 + \left(\frac{15x_n^4+60hx_n^3+60h^2x_n^2-8h^4}{360h^3}\right)x - \left(\frac{x_n^3+3hx_n^2+2h^2x_n}{12h^3}\right)x^2 + \left(\frac{3x_n^2+6hx_n+2h^2}{36h^3}\right)x^3 - \left(\frac{x_n+h}{24h^3}\right)x^4 +$$

$$\left(\frac{1}{120h^3}\right)x^5$$

$$\beta_3(x) = -\left(\frac{-3(x-x_n)^5+15h(x-x_n)^4-20h^2(x-x_n)^3+8h^4(x-x_n)}{360h^3}\right) \quad (20)$$

Substituting the values of ($\alpha_{i's}$ and $\beta_{i's}$) from equation $(15-20)$ respectively into equation (14) gives rise to (21)

$$
\begin{aligned}
y(x) &= \left(\frac{x_n+h}{h}-\frac{\mathrm{x}}{\mathrm{h}}\right)y_n+\left(\frac{-x_n}{h}+\frac{x}{h}\right)y_{n+1} \\
&-\left(\frac{3(x-x_n)^5-30h(x-x_n)^4+110h^2(x-x_n)^3-180h^3(x-x_n)^2+97h^4(x-x_n)}{360h^3}\right)f_n \\
&-\left(\frac{-3(x-x_n)^5+25h(x-x_n)^4-60h^2(x-x_n)^3+38h^4(x-x_n)}{120h^3}\right)f_{n+1} \\
&+\left(\frac{-3(x-x_n)^5+20h(x-x_n)^4-30h^2(x-x_n)^3+13h^4(x-x_n)}{120h^3}\right)f_{n+2} \\
&-\left(\frac{-3(x-x_n)^5+15h(x-x_n)^4-20h^2(x-x_n)^3+8h^4(x-x_n)}{360h^3}\right)f_{n+3}
\end{aligned} \tag{21}
$$

Evaluating (21) at non-interpolating points i.e $x=x_n+2h, y(x)=y_{n+2}$ and $x=x_n+3h, y(x)=y_{n+3}$ (recall the interpolating points are $m=0$ and 1) will aid to obtain the first two schemes which are part of the propose method

$$y_{n+2}+y_n-2y_{n+1}=\frac{h^2}{12}(f_n+10f_{n+1}+f_{n+2}) \tag{22}$$

$$y_{n+3}+2y_n-3y_{n+1}=\frac{h^2}{12}(2f_n+21f_{n+1}+12f_{n+2}+f_{n+3}) \tag{23}$$

Differentiating $\alpha_0\ (x), \alpha_1\ (x), \beta_0\ (x), \beta_1\ (x), \beta_2\ (x)$ and $\beta_3\ (x)$ respectively will aid in determining the remaining schemes that will complete the method

$$
\left.
\begin{aligned}
\alpha'_0(x) &= -\frac{1}{h} \\
\alpha'_1(x) &= \frac{1}{h} \\
\beta'_0(x) &= -\left(\frac{15(x-x_n)^4-120h(x-x_n)^3+330h^2(x-x_n)^2-360h^3(x-x_n)+97h^4}{360h^3}\right) \\
\beta'_1(x) &= -\left(\frac{-15(x-x_n)^4+100h(x-x_n)^3-180h^2(x-x_n)^2+38h^4}{120h^3}\right) \\
\beta'_2(x) &= \left(\frac{-15(x-x_n)^4+80h(x-x_n)^3-90h^2(x-x_n)^2+13h^4}{120h^3}\right) \\
\beta'_3(x) &= -\left(\frac{-15(x-x_n)^4+60h(x-x_n)^3-60h^2(x-x_n)^2+8h^4}{360h^3}\right)
\end{aligned}
\right\} \tag{24}
$$

Substituting (24) in the derivative of (14) will generate other schemes

$$y'(x)=\alpha'_0(x)y_n+\alpha'_1(x)y_{n+1}+h^2(\beta'_0(x)f_n+\beta'_1(x)f_{n+1}+\beta'_2(x)f_{n+2}+\beta'_3(x)f_{n+3}) \tag{25}$$

$$
\begin{aligned}
y'(x) &= -\frac{1}{h}y_n+\frac{1}{h}y_{n+1}-\left(\frac{15(x-x_n)^4-120h(x-x_n)^3+330h^2(x-x_n)^2-360h^3(x-x_n)+97h^4}{360h^3}\right)f_n- \\
&\left(\frac{-15(x-x_n)^4+100h(x-x_n)^3-180h^2(x-x_n)^2+38h^4}{120h^3}\right)f_{n+1}+ \\
&\left(\frac{-15(x-x_n)^4+80h(x-x_n)^3-90h^2(x-x_n)^2+13h^4}{120h^3}\right)f_{n+2}-\left(\frac{-15(x-x_n)^4+60h(x-x_n)^3-60h^2(x-x_n)^2+8h^4}{360h^3}\right)f_{n+3}
\end{aligned} \tag{26}
$$

Therefore, evaluating (26) at the point 0,1,2 and 3 gives

$$y_{n+1} = \frac{h^2}{360}(97f_n + 114f_{n+1} - 39f_{n+2} + 8f_{n+3}) + y_n + hy'_n$$
$$hy'_{n+1} - y_{n+1} = \frac{h^2}{360}(38f_n + 171f_{n+1} - 36f_{n+2} + 7f_{n+3}) - y_n$$
$$hy'_{n+2} - y_{n+1} = \frac{h^2}{360}(23f_n + 366f_{n+1} + 159f_{n+2} - 8f_{n+3}) - y_n$$
$$hy'_{n+3} - y_{n+1} = \frac{h^2}{360}(38f_n + 291f_{n+1} + 444f_{n+2} + 127f_{n+3}) - y_n$$
$$y_{n+2} - 2y_{n+1} = \frac{h^2}{12}(f_n + 10f_{n+1} + f_{n+2}) - y_n$$
$$y_{n+3} - 3y_{n+1} = \frac{h^2}{12}(2f_n + 21f_{n+1} + 12f_{n+2} + f_{n+3}) - 2y_n \qquad (27)$$

Putting the values of (27) in matrix form, we have

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{h^2}{360}(97f_n + 114f_{n+1} - 39f_{n+2} + 8f_{n+3}) + y_n + hy'_n \\ -1 & 0 & 0 & h & 0 & 0 & \frac{h^2}{360}(38f_n + 171f_{n+1} - 36f_{n+2} + 7f_{n+3}) - y_n \\ -1 & 0 & 0 & 0 & h & 0 & \frac{h^2}{360}(23f_n + 366f_{n+1} + 159f_{n+2} - 8f_{n+3}) - y_n \\ -1 & 0 & 0 & 0 & 0 & h & \frac{h^2}{360}(38f_n + 291f_{n+1} + 444f_{n+2} + 127f_{n+3}) - y_n \\ -2 & 1 & 0 & 0 & 0 & 0 & \frac{h^2}{12}(f_n + 10f_{n+1} + f_{n+2}) - y_n \\ -3 & 0 & 1 & 0 & 0 & 0 & \frac{h^2}{12}(2f_n + 21f_{n+1} + 12f_{n+2} + f_{n+3}) - 2y_n \end{bmatrix}$$

(27)

Then matrix B is solved by Gauss - Jordan Elimination method using Maple17 and we have:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{h^2}{360}(97f_n + 114f_{n+1} - 39f_{n+2} + 8f_{n+3}) + y_n + hy'_n \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{h^2}{45}(28f_n + 66f_{n+1} - 6f_{n+2} + 2f_{n+3}) + y_n + 2hy'_n \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{h^2}{40}(39f_n + 108f_{n+1} + 27f_{n+2} + 6f_{n+3}) + y_n + 3hy'_n \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{h}{24}(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}) + y'_n \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{h}{3}(f_n + 4f_{n+1} + f_{n+2}) + y'_n \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{h}{8}(3f_n + 9f_{n+1} + 9f_{n+2} + f_{n+3}) + y'_n \end{bmatrix}$$

(28)

Therefore, the solution is

$$\left.\begin{aligned} y_{n+1} &= \frac{h^2}{360}(97f_n + 114f_{n+1} - 39f_{n+2} + 8f_{n+3}) + y_n + hy'_n \\ y_{n+2} &= \frac{h^2}{45}(28f_n + 66f_{n+1} - 6f_{n+2} + 2f_{n+3}) + y_n + 2hy'_n \\ y_{n+3} &= \frac{h^2}{40}(39f_n + 108f_{n+1} + 27f_{n+2} + 6f_{n+3}) + y_n + 3hy'_n \\ y'_{n+1} &= \frac{h}{24}(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}) + y'_n \\ y'_{n+2} &= \frac{h}{3}(f_n + 4f_{n+1} + f_{n+2}) + y'_n \\ y'_{n+3} &= \frac{h}{8}(3f_n + 9f_{n+1} + 9f_{n+2} + f_{n+3}) + y'_n \end{aligned}\right\} \quad (29)$$

3. Results and Discussions

3.1 Order and Error Constant

To determine the order and error constants of the block methods develop, we consider the general linear multi-step method of the form:

$$\sum_{i=0}^{k} \alpha_i \ y_{n+i} = h^2 \left\{ \sum_{i=0}^{k} \beta_i \ f_{n+i} \right\} \quad (30)$$

(Fatunla S. O, 1991)
We consider the definition of associated linear difference operator L defined as;

$$L[y(x);h] = \sum_{i=0}^{k} [\alpha_i \ y(x+ih) - h^2 \beta_i \, y''(x+ih)] \quad (31)$$

((Fatunla S. O, 1991)
Where $y(x)$ is an arbitrary function which is continuously differentiable many times on interval$[a,b]$. Expanding (31) using Taylor series about $y(x)$ and if we gather the coefficients of power of h, we obtain

$$L[y(x);h] = c_0 \, y(x) + c_1 \, hy'(x) + c_2 \, h^2 y''(x) + \cdots + c_q \, h^q y^q(x) + 0(h^{q+1}) \quad (32)$$

whose coefficients c_q , $q = 0,1,2, \ldots \ldots ....$ are constants and given as:
Thus (32) is said to be order p if and only if $c_0 = c_1 = c_2 = \cdots c_{p+1} = 0$ and $c_{p+2} \neq 0$, that is (if c_{p+2} does not vanish) and the error term (constant) is said to be the coefficient of order $(p+2)$, which will be the method that will be used in obtaining the order and error constants of the schemes obtained.
This implies that the block method has order $p = (4,4,4)^T$ with error constants $c_{p+2} = \left(-\frac{1}{240}, -\frac{1}{80}, -\frac{9}{160}\right)^T$

3.2 Zero-Stability

A linear multistep method is said to be zero stable provided $R_{ij} = 1(1)k$ of the first characteristics polynomial ρ(λ) specifies as $\rho(\lambda) = det\left[\lambda I - A_1^{(1)}\right]$

Substituting the values of λI and $A_1^{(1)}$ in the function above, gives

$$\rho(\lambda) = det\left[\lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & -1 & 3\\0 & -2 & 6\\0 & -1 & 3\end{bmatrix}\right] = det\left[\begin{bmatrix}\lambda & 0 & 0\\0 & \lambda & 0\\0 & 0 & \lambda\end{bmatrix} - \begin{bmatrix}0 & -1 & 3\\0 & -2 & 6\\0 & -1 & 3\end{bmatrix}\right]$$

$$= det\left[\begin{bmatrix}\lambda & 1 & -3\\0 & \lambda+2 & -6\\0 & 1 & \lambda-3\end{bmatrix}\right] = \begin{vmatrix}\lambda & 1 & -3\\0 & \lambda+2 & -6\\0 & 1 & \lambda-3\end{vmatrix}$$

Solving the above determinant yields

$$\rho(\lambda) = \lambda[(\lambda+2)(\lambda-3)+6] - 1[0+0] - 3[0+0] = \lambda[\lambda^2 - \lambda - 6 + 6] = \lambda(\lambda^2 - \lambda)$$

$\rho(\lambda) = \lambda^2(\lambda - 1)$, which implies, $\lambda_1 = \lambda_2 = 0\ or\ \lambda_3 = 1$ (33)

From equation (33) its shows that the block method is zero stable and is also consistent as its order $(4,4,4)^T > 1$, thus, the scheme is convergent. The following problems are considered;

Problem 1:

$$y" + \lambda^2 y = 0$$

with the initial conditions;

$$y(0) = 1, y'(0) = 2, h = 0.01$$

With $\lambda = 2$,whose exact solution; $y(x) = \sin(2x) + \cos(2x)$

Source: (Awoyemi and Adebile, 2011)

Problem 2:

$y'' - y' = 0$

with the initial conditions; $y(0) = 0, y'(0) = -1, h = 0.1$

whose exact solution is $y(x) = 1 - e^x$

Source: (Adeniran et al, 2015)

X-value	Exact value	Error in proposed method	Error in Awoyemi et al (2011)
0.01	1.01979867335991	9.601E-13	2.6577E-11
0.02	1.03918944084761	2.190E-12	8.471E-10
0.03	1.05816454641464	3.690E-12	6.4146E-09
0.04	1.07671640027179	4.614E-10	6.7071E-09
0.05	1.09483758192485	3.567E-10	7.1209E-09
0.06	1.11252084314278	2.522E-10	7.6530E-09
0.07	1.14654548998987	2.115E-10	8.3601E-09
0.08	1.16287326621394	2.678E-10	9.0592E-09
0.09	1.17873590863630	7.057E-10	9.9268E-09
0.10	1.12975911085687	3.090E-10	1.0899E-08

Table1: Comparison of error for problem 1

4.0 Conclusion

This work demonstrated a successful application of 3-step block method to solve a second order differential equations directly without reducing it to first order ODEs. The proposed block methods are self-starting and does not required any predictors to estimate in the integrators. The order and error constants of the discrete scheme constitute zero-stable block integrator of order$(4,4,4)^T$. All the proposed block methods are more accurate when compared with exact solution.

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X-value	Exact value	Error in proposed method	Error in Adeniran et al. (2015)
0.01	-0.10517093561	9.601E-13	2.220E-08
0.02	-0.22140280068	2.190E-12	1.250E-07
0.03	-0.34985888231	3.690E-12	3.250E-07
0.04	-0.49182484977	4.614E-10	6.424E-06
0.05	-0.64872151609	3.567E-10	1.099E-06
0.06	-0.82211915495	2.522E-10	1.721E-06
0.07	-1.01375323952	2.115E-10	2.538E-06
0.08	-1.22554166636	2.678E-10	3.583E-06
0.09	-1.45960408486	7.057E-10	4.896E-06
0.10	-1.71828313929	3.090E-10	6.522E-06

Table 2: Comparison of error for problem 2

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