ON THE NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY OF PAT-JIYOR MODEL

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Abstract
This work applies the Karush-Kuhn-Tucker (KKT) Necessary and Sufficient condition on the Pat-JiYor Model formulated to obtain the optimum Revenue for the Production of Ethanol from Biomass. The result shows that the principal minor determinants are negative definite which implies that the Revenue was a maximum value.
Keywords: KKT necessary and sufficient condition, Revenue, optimum,

Introduction

Ethanol is gotten mainly from grains like corn, millets and other sources like cassava, sugarcane. These crops are sources of food for humans and feed for animals. To reduces cost of production of ethanol and burden on food resources for fuel, Biomass such as municipal waste, forest residues, agricultural waste, woody materials, organic by – products are also used for ethanol production (Egwim *et al*., 2015).

Lagrange multiplier method is a technique for finding a maximum or minimum of a function $F(x,y)$ subject to a constraint of the given form $G(x, y) = 0$ (Salih, 2013). Pat-JiYor Model is a new model derived as a modification of Cobb-Doglass model to find the optimal revenue for ethanol production from Banana trunk biomass (Nyor *et al*., 2018). The model is coined after the names of the authors Patience Evans, Jiya Mohammed and Nyor Ngutor (Patience-Jiya Nyor). This work seeks to establish whether or not the optimal revenue of four thousand five hundred and thirty naira which was obtained using Pat-JiYor model for the production of ethanol is either maximum or minimum or neither using the Karush-Kuhn-Tucker (KKT) conditions

The KKT condition is a mathematical optimization first order necessary condition also known as Kuhn-Tuker conditions for the optimal solution of a nonlinear programming problem. In nonlinear programming KKT theorem also referred to as saddle-point theorem uses the method of Lagrange multipliers on the equality constrained problem to convert the constrained optimization problem to an unconstrained problem to obtain a local and or global maximum (minimum) in the domain. According to Taha (2010), the KKT condition provide the most unifying theory for all nonlinear programing problems.

KKT condition can be applied by obtaining the second order partial derivative of the interior maximum for each of the variable which is represented as a matrix. The solution will be a maximum, minimum or a saddle point using the method of principal minor determinant which is the necessary and sufficient conditions for optimality (Taha, 2010).

Methodology

To develop the general Karush-Kuhn Tucker (KKT) necessary conditions for determining the stationary points which are also sufficient under certain rules, Taha (2010) considers the problem of the form,

$$Optimize:\ Z = f(X)$$

$$Subject\ to: g(X) \leq 0$$

The inequality constraint may be converted into equations by using nonnegative slack variables let S_i^2 (≥ 0) be the slack quantity added to the ith constraint $g_i(X) \leq 0$ and define.

$$S = (S_{1,}S_2, \ldots, S_m)^T, S^2 = (S_1^2, S_2^2, \ldots, S_m^2)^T$$

Where m is the number of inequality constraints. The Lagrangean function is thus given by,

$$L(S, X, \lambda) = f(X) - \lambda\,[g(X) + S^2]$$

Given the constraints,

$$g(X) \leq 0$$

A necessary condition for optimality is that λ be nonnegative (nonpositive) for maximization (minimization) problems. This result is justified by noting that the vector λ measures the rate of variation of f with respect to g…that is,

$$\lambda = \frac{\partial f}{\partial g}$$

In maximization case, as the right-hand side of the constraint $\mathrm{g(X)} \leq 0$ increases from 0 to the vector ∂g, the solution space becomes less constraint and hence f cannot decrease, meaning that $\lambda \geq 0$. Similarly, for minimization, as the right-hand side of the constraints increases, f cannot increase, which implies that $\lambda \leq 0$. If the care equalities, that is, $g(X) = 0$, then λ becomes unrestricted in sign.

The restrictions on λ holds as part of the KKT necessary conditions. The other conditions are developed as follows,

Taking the partial derivatives of L with respect to X, S, and λ, we obtain,

$$\frac{\partial L}{\partial X} = \nabla f(X) - \lambda \forall \nabla g(X) = 0$$

$$\frac{\partial L}{\partial S_i} = -2\lambda_i\, S_i = 0, i = 1,2,3, \ldots, m$$

$$\frac{\partial L}{\partial \lambda} = -(g(X) + S^2) = 0$$

These sets of equations reveal the results as follows (Taha, 2010; Rardin, 1998):

1. If $\lambda_i = 0$, then S_i^2 this means that the corresponding resource is scares, and hence it is consumed completely (inequality constraint).
2. If $S_i^2 > 0$, then $\lambda_i = 0. this\ means\ resource$, i is not scarce and, consequently it has no effect on the value of $f\left(i.e. \lambda_i = \frac{\partial y}{\partial x} = 0\right)$

   From the second and third sets of equations, we obtain,

$$\lambda g(X) = 0 \qquad\qquad i = 1, 2\,, \ldots, m$$

This new condition essentially repeats the forgoing arguments, because if $\lambda > 0$, $then\ g(X) = 0$ or $S_i^2 = 0$ and if if $then\ g(X) < 0$ and $S_i^2 > 0$ and $\lambda = 0$,

According to Taha (2010) the KKT necessary conditions for maximization problem are summarized as:

$$\lambda \geq 0$$

$$\nabla f(X) - \lambda\, \nabla g(X) = 0$$

$$\lambda_i g_i(X) = 0, i = 1,2, \ldots, m$$

$$g(X) \leq 0$$

These conditions apply to minimization case as well, except that λ must be non-positive. In both maximization and minimization, the Lagrange multipliers corresponding to equality constraints are restricted in sign.

On the Sufficiency of the KKT Conditions. The Kuhn-Tucker necessary conditions are also sufficient if the objective function and the solution space satisfy specific conditions (Taha, 2010). These conditions are summarized in Table 3.1.

It is simpler to verify that a function is convex or concave than to prove that a solution space is a convex set. For this reason, we provide a list of conditions that are easier to apply in practice in the sense that the convexity of the solution space can be established by checking the convexity or concavity of the constraint functions. To provide these conditions, we define the generalized nonlinear problems as,

Maximize or minimize $z = f(X)$

subject to $g_i(X) \leq 0,\ i = 1, 2, \ldots, r$

$g_i(X) \geq 0,\ i = r + 1, \ldots, p$

$g_i(X) = 0,\ i = P + 1, \ldots, m$

$$L(X, S, \lambda) = f(X) - \sum_{i=1}^{t} \lambda_i \left(g_i(X) + S_i^2\right) - \sum_{i=r+1}^{p} \lambda_i \left(g_i(X) - S_i^2\right) - \sum_{i=p+1}^{m} \lambda_i \left(g_i(X)\right)$$

(Bazarra *et al.*, 1993)

where λ_i is the Lagrangean multiplier associated with constraint *i.* The conditions for establishing the sufficiency of the KKT conditions are summarized in Table 3.2.

The conditions in Table 3.2 represent only a subset of the conditions in Table3.1 because a solution space may be convex without satisfying the conditions in Table 3.2.

Table 3.1: Required Conditions

Sense of Optimization	Objective Function	Solution Space
Minimization	Concave	Convex Set
Maximization	Convex	Convex Set

Table 3.2: Required Conditions

Sense of optimization	F(x)	Required Conditions g(x)	λ	
maximization	**concave**	**Convex concave Linear**	**≥ 0 ≤ 0 unrestricted**	**(1 ≤ i ≤0) (r +1 ≤ i ≤p) (p + 1 ≤ i ≤ m)**
minimization	**convex**	**Convex concave Linear**	**≤ 0 ≥ 0 unrestricted**	**(1 ≤ i ≤0) (r +1 ≤ i ≤p) (p + 1 ≤ i ≤ m)**

Table 3.2 is valid because the given conditions yield a concave Langrangean function L (X, S, λ) in case of maximization and a Convex L(X, S, λ) in case of minimization. This result is verified by noticing that if g(X) is convex, then λg(X) is convex, if $\lambda_i \geq 0$ and concave if $\lambda_i \leq 0$. Similar interpretations can be established for all the remaining conditions. A linear function is both convex and concave. If a function f is concave, then (-f) is convex, and vice versa (Beightler *et al.* 1979; Taha, 2010).

Given a problem of the form

$$\min f(x) \tag{1}$$

$$s.t \quad g_i - b_i \geq 0 \quad i = 1, \dots, k \tag{2}$$
$$g_i - b_i = 0 \quad i = k+1, \dots, m \tag{3}$$

There are four KKT necessary conditions model optimality

1. Feasibility
$$g_{i\,(x)} - b_i \quad is\ feasible\ (applies\ to\ 2, 3)$$

2. No direction which improves objective and feasibility,
$$\nabla f(x)^* - \sum_{i=1}^{m} \lambda_i^* \; g_i(x^*) = 0 \qquad (applies\ to\ n1, 2, 3)$$

3. Complimentary slackness,
$$\lambda_i^* \left(g_i(x^*) - b_i\right) = 0 \quad i = 1, \dots, k \;\; (Applies\ to\ 2)$$
4. Positive Langrange multipliers
$$\lambda_i \geq 0 \qquad i = 1, \dots k \qquad (Applies\ to\ 2)$$

Application of KKT to Pat-JiYor Revenue Model.

Given the Pat-JiYor production model (Nyor et al., 2018),
$$P(h, r, s, t) = kh^a r^b s^c t^d \tag{4}$$
$$\text{Subject to: } 120\text{h} + 50\text{r} + 40\text{s} + 70\text{t} = 4000 \tag{5}$$
$$\forall \quad 0 < a < 1,\; 0 < b < 1,\; 0 < c < 1, \qquad 0 < d < 1$$
$$\forall\, h, r, s, t > 0$$
$$where, \qquad P = Revenue, k = constnt, h = hours\ of\ labour,$$
$$r = biomass. s = residue, t = glucose$$

By heuristic approach, optimal revenue was obtained using the relationship,
$$max: P(h, r, s, t) = 200\, h^{\frac{3}{4}} r^{\frac{1}{2}} s^{\frac{1}{6}} t^{\frac{1}{12}} \tag{6}$$
$$\text{Subject to: } 120\text{h} + 50\text{r} + 40\text{s} + 70\text{t} = 4000 \tag{7}$$

1st KKT condition requires the equality constraints to be in residing form,
$$120\text{h} + 50\text{r} + 40\text{s} + 70\text{t} - 4000 = 0 \tag{8}$$

2nd KKT condition requires the application of Lagrange multiplier
$$L\,(h, r, s, t) = 200\, h^{\frac{3}{4}} r^{\frac{1}{2}} s^{\frac{1}{6}} t^{\frac{1}{12}} \quad - \lambda\,(\,120\text{h} + 50\text{r} + 40\text{s} + 70\text{t} - 4000) = 0 \tag{9}$$

Differentiating equation (9) partially with respect to h, r, s, t and λ, we obtain,
$$L_1\,(h) = \frac{600}{4}\, h^{\frac{-1}{4}} r^{\frac{1}{2}} s^{\frac{1}{6}} t^{\frac{1}{12}} \quad - 120\lambda = 0 \tag{10}$$
$$L_2\,(r) = \frac{200}{2}\, h^{\frac{3}{4}} r^{\frac{-1}{2}} s^{\frac{1}{6}} t^{\frac{1}{12}} \quad - 500\lambda = 0 \tag{11}$$
$$L_3\,(s) = \frac{200}{6}\, h^{\frac{3}{4}} r^{\frac{1}{2}} s^{\frac{-5}{6}} t^{\frac{1}{12}} \quad - 422\lambda = 0 \tag{12}$$
$$L_4\,(t) = \frac{200}{12}\, h^{\frac{3}{4}} r^{\frac{1}{2}} s^{\frac{1}{6}} t^{\frac{-11}{12}} \quad - 383\lambda = 0 \tag{13}$$
$$L_5\,(\lambda) = -120\text{h} - 500\text{r} - 422\text{s} - 383\text{t} = 0 \tag{14}$$

The 3ed KKT condition do not apply to the problem, since there is no inequality constraint. Inputting equations (10), (11), (12), (13) and (14), into MAPLE17 Software, we obtain
$$h = 16.67\,, r = 2.67\,, s = 11.11\,, t = 3.17,\ \lambda = 1.6$$

KKT condition of $\lambda^* \geq 0 \;\; (\lambda = 1.6)$ shows that the solution is optimal. If λ^* is negative it means it is not optimum and needs further investigation.

The 4th KKT condition is satisfied since λ^* is positive, this means the solution is optimum.

Again, it is required to know if the optimum solution is a maximum or minimum which leads to the sufficient (second order condition) which put restrictions on the Hessian denoted by $H_L(*)$

$$H_L = \begin{pmatrix} \frac{\partial^2 L_1}{\partial h^2} & \frac{\partial^2 L_1}{\partial hr} & \frac{\partial^2 L_1}{\partial hs} & \frac{\partial^2 L_1}{\partial ht} & \frac{\partial^2 L_1}{\partial h\lambda} \\ \frac{\partial^2 L_2}{\partial rh} & \frac{\partial^2 L_2}{\partial r^2} & \frac{\partial^2 L_2}{\partial rs} & \frac{\partial^2 L_2}{\partial rt} & \frac{\partial^2 L_2}{\partial r\lambda} \\ \frac{\partial^2 L_3}{\partial sh} & \frac{\partial^2 L_3}{\partial sr} & \frac{\partial^2 L_3}{\partial s^2} & \frac{\partial^2 L_3}{\partial st} & \frac{\partial^2 L_3}{\partial s\lambda} \\ \frac{\partial^2 L_4}{\partial th} & \frac{\partial^2 L_4}{\partial tr} & \frac{\partial^2 L_4}{\partial ts} & \frac{\partial^2 L_4}{\partial t^2} & \frac{\partial^2 L_4}{\partial t\lambda} \\ \frac{\partial^2 L_5}{\partial \lambda h} & \frac{\partial^2 L_5}{\partial \lambda r} & \frac{\partial^2 L_5}{\partial \lambda s} & \frac{\partial^2 L_5}{\partial \lambda t} & \frac{\partial^2 L_5}{\partial \lambda^2} \end{pmatrix} \quad (15)$$

$$H_L = \begin{pmatrix} \frac{-600}{16} h^{\frac{-5}{4}} r^{\frac{1}{2}} s^{\frac{1}{6}} t^{\frac{1}{12}} & \frac{600}{8} h^{\frac{-1}{4}} r^{\frac{-1}{2}} s^{\frac{1}{6}} t^{\frac{1}{12}} & \frac{600}{24} h^{\frac{-1}{4}} r^{\frac{1}{2}} s^{\frac{-5}{6}} t^{\frac{1}{12}} & \frac{600}{48} h^{\frac{-1}{4}} r^{\frac{1}{2}} s^{\frac{1}{6}} t^{\frac{-11}{12}} & -120 \\ \frac{600}{8} h^{\frac{-1}{4}} r^{\frac{-1}{2}} s^{\frac{1}{6}} t^{\frac{1}{12}} & \frac{-200}{4} h^{\frac{3}{4}} r^{\frac{-3}{2}} s^{\frac{1}{6}} t^{\frac{1}{12}} & \frac{200}{12} h^{\frac{3}{4}} r^{\frac{-1}{2}} s^{\frac{-5}{6}} t^{\frac{1}{12}} & \frac{-600}{24} h^{\frac{3}{4}} r^{\frac{-1}{2}} s^{\frac{1}{6}} t^{\frac{-11}{12}} & -500 \\ \frac{600}{24} h^{\frac{-1}{4}} r^{\frac{1}{2}} s^{\frac{-5}{6}} t^{\frac{1}{12}} & \frac{200}{12} h^{\frac{3}{4}} r^{\frac{-1}{2}} s^{\frac{-5}{6}} t^{\frac{1}{12}} & \frac{-1000}{36} h^{\frac{3}{4}} r^{\frac{1}{2}} s^{\frac{-11}{6}} t^{\frac{1}{12}} & \frac{200}{72} h^{\frac{3}{4}} r^{\frac{1}{2}} s^{\frac{-5}{6}} t^{\frac{-11}{12}} & -422 \\ \frac{600}{48} h^{\frac{-1}{4}} r^{\frac{1}{2}} s^{\frac{1}{6}} t^{\frac{-11}{12}} & \frac{200}{24} h^{\frac{3}{4}} r^{\frac{-1}{2}} s^{\frac{1}{6}} t^{\frac{-11}{12}} & \frac{200}{72} h^{\frac{3}{4}} r^{\frac{1}{2}} s^{\frac{-5}{6}} t^{\frac{-11}{12}} & \frac{-2200}{144} h^{\frac{3}{4}} r^{\frac{1}{2}} s^{\frac{1}{6}} t^{\frac{-23}{12}} & -383 \\ -120 & -500 & -422 & -383 & 0 \end{pmatrix} \quad (16)$$

Results and Discussion

The result of the second order partial derivative is give as

$$H_L = \begin{pmatrix} -2.991630923 & 37.35617039 & 2.992528502 & 5.244004990 & -120 \\ 37.35617039 & -155.4874806 & 12.45579279 & 21.82710691 & -500 \\ 2.992528502 & 12.45579279 & -4.989043922 & 1.748526117 & -422 \\ 5.244004990 & 21.82710691 & 1.748526117 & -67.40926708 & -383 \\ -120 & -500 & -422 & -383 & 0 \end{pmatrix}$$

The 1st principal minor determinant = - 2.9916303923

The 2nd principal minor determinant is $\begin{pmatrix} -2.991630923 & 37.35617039 \\ 37.35617039 & -155.4874806 \end{pmatrix} = -930.3223109$

The 3rd principal minor determinant is

$$\begin{pmatrix} -2.991630923 & 37.35617039 & 2.992528502 \\ 37.35617039 & -155.4874806 & 12.45579279 \\ 2.992528502 & 12.45579279 & -4.989043922 \end{pmatrix} = 9282.837735$$

The 4rd principal minor determinant is

$$\begin{pmatrix} -2.991630923 & 37.35617039 & 2.992528502 & 5.244004990 \\ 37.35617039 & -155.4874806 & 12.45579279 & 21.82710691 \\ 2.992528502 & 12.45579279 & -4.989043922 & 1.748526117 \\ 5.244004990 & 21.82710691 & 1.748526117 & -67.40926708 \end{pmatrix}$$

$$= -77.23455996810^{5}$$

The 5th principal minor determinant is

$$= \begin{pmatrix} -2.991630923 & 37.35617039 & 2.992528502 & 5.244004990 & -120 \\ 37.35617039 & -155.4874806 & 12.45579279 & 21.82710691 & -500 \\ 2.992528502 & 12.45579279 & -4.989043922 & 1.748526117 & -422 \\ 5.244004990 & 21.82710691 & 1.748526117 & -67.40926708 & -383 \\ -120 & -500 & -422 & -383 & 0 \end{pmatrix}$$

$$= 2.095965139$$

The hessian matrix obtained is a Symmetric matrix The Principal Minor determinates are $-2.9916303923,\ -930.3223109, 9282.837735, -77.23455996810^5\ and\ \ 2.095965139$, respectively.
According to Taha (2003), Since the principal minor determinants are negative definite, this indicates the maximum point.

Conclusion
The negative definite result obtained shows that the optimum solution by Pat-JiYor is actually a maximum revenue that can be obtained under that market condition.
The present study concludes that the Pat-JiYor fulfills all the conditions of KKT, it is therefore a good model for revenue optimization.

References
Bazarra, M., Shrali, H., & Shetty, C. (1993). *Nonlinear Programming Theory and Algorithms,* 2nd ed., Wiley, New York.

Beightler, C., Phillips, D., & Wilde, D. (1979). *Foundations of Optimization,* 2nd ed., Prentice Hall, New Jersey.

Egwim, E. C., Shittu, K. O., & Komolafe, D. (2015). Microwave – alkaline assisted pretreatment of banana trunk for bioethanol production. *Journal of Energy and Power Engineering,* 9, 705 – 713.

Nyor, N., Evans, P. O., Jiya, M., & Yisa, J. (2018). A lagrange revenue model for ethanol production from biomass. *Transactions of the Nigerian Association of Mathematical Physics*, 7, 225 – 228.

Rardin, R. (1998.) *Optimization in Operations Research,* Prentice Hall, New Jersey.

Salih, A. (2013). Department of Aerospace Engineering Indian Institute of Space Science and Technology, Thiruvananthapuram.

Taha, H. A. (2010) *Operations Research: An Introduction* 8th ed. Pearson and prentice Hallm upper saddle River, New Jersey. ISBN 0-13-188923·0