A MATHEMATICAL MODEL FOR CONTAMINANT TRANSPORT IN AN UNCONFINED AQUIFER SYSTEM

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Abstract
The analysis of contaminant transport into an aquifer system, showing the behavior of contaminants over a time period is of paramount importance in the study of geological behavior of the aquifer system. In this paper a mathematical model for Contaminant Transport in an Unconfined Aquifer was formulated using a Laplace transform. The governing equation for solute transport given by Kumar (20014) was used for the formulation. The analysis of contaminant transport in an aquifer system, showing the behavior of contaminants for different values of Diffusive transport into the unconfined layer (α) for $0 < t \leq 7$ was modeled, the diffusion within the layer (y), and the size of the Aquifer (b) kept constant. For a uniform source of contamination at it $c''(x)$ was observed that for different values of a, b and y the level of contamination reduces over the domain.

Keywords: Contaminant transport, unconfined aquifer, dispersion, advection, piezometric and head

Introduction

Water is therefore the most essential element for man's well-being, social and economic progress. Groundwater offers the most abundant source of water to man. It is the cheapest and the most constant in quality and quantity (Olasehinde, 2014). It is observed that in many developing countries, groundwater plays a major source of support for domestic needs and irrigation purposes (Thangarajan, 2014). Water shortages occur quite often in many areas of the world, calling for optimal management of both surface and groundwater resources (Helmut 2014, Jacques, 2014). Groundwater quality is usually better, since they are naturally more protected, once polluted, their restoration is more difficult, calling for optimal control of groundwater contamination (Amlan & Bithin, 2015; Fetter, 2014). Sagei et al. (2015) considered a fractured confined porous aquifer, and came up with a modeled solute equation, that analysed the effect of non-Fickian diffusion into surrounding rocks.

The aim of this paper to formulate a mathematical model that can be used to simulate a solute transport and analyze the movement of contaminants in an unconfined aquifer system.

Model Formulation

We considered a governing equation for solute transport given by Kumar (2014) as:

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x_i}(CV_i) + \frac{\partial}{\partial x_i}\left(D_{ij}\frac{\partial C}{\partial x_j}\right) + R_c \qquad i, j = 1,2,3 \tag{2.1}$$

$$V_i = \frac{K_{ij}}{\theta}\frac{\partial h}{\partial x_i} \tag{2.2}$$

where;

D_{ij} = Coefficient of Hydrodynamic Dispersion,
C = Concentration of the Solute in the Source or Sink Fluid
R_c = Sources or Sinks
V_i = Seepage Velocity

K_{ij} = Hydraulic Conductivity
h = Hydraulic Head
x_i = Coordinate system

The initial condition (specification of the Concentration distribution of Solute at initial time t = 0), can be written as;

$$C(x) = C^U(x) \qquad x \in \Omega \qquad (2.3)$$

where;

$C^U(x)$ indicates a known Concentration distribution over the domain of interest (Ω).

$C^u = 1$ indicates a Uniform Source of Contamination at $X = 0$

Here we consider an unconfined in the formula tion of our model. Three regions were considered; the upper layer (porous layer1), middle layer (Aquifer layer) and the lower layer (porous layer 2), as shown in figure 2.1

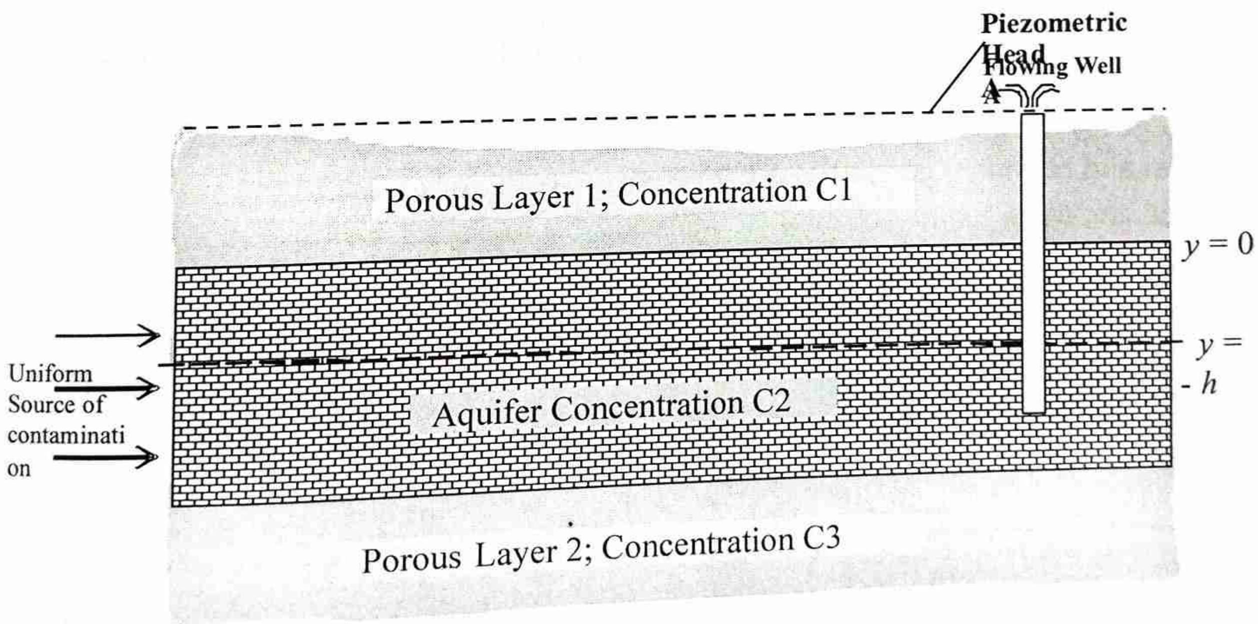

Figure 2.1 Diagram of an unconfined Aquifer for the model formulation

The partial differential equation describing the contaminant transport in the upper layer is given as;

$$\frac{\partial c_1}{\partial \tau} = D_1 \frac{\partial}{\partial y}\left[\frac{\partial^\alpha c_i}{\partial y^\alpha}\right] \qquad 0 < y < \infty,\ \tau > 0 \qquad (2.4)$$

where

D_1	=	Effective Diffusivity of Porous layer
c	=	contaminant concentration

Similarly, the partial differential equation describing the contaminant transport in the aquifer is given as;

$$\frac{\partial c_2}{\partial \tau} + \beta \frac{\partial^\gamma c_2}{\partial \tau^\gamma} = \overline{D}_2 \frac{\partial}{\partial y}\left(\frac{\partial^\lambda c_{12}}{\partial y^\lambda}\right) + D_2 \frac{\partial}{\partial x}\left(\frac{\partial^\lambda c_2}{\partial x^\lambda}\right) - v\frac{\partial c_2}{\partial x} \tag{2.5}$$

$$-h < y < 0,\ 0 < x < \infty,\ \tau > 0$$

D_2 and $\overline{D}_2$ = Effective Diffusivities, in the aquifer in the x and y-direction, respectively;

v	=	Fluid Velocity
τ	=	Time
β	=	Capacity Coefficient
$\frac{\partial^\gamma c_2}{\partial \tau^\gamma}$	=	Fractional-in-time Derivatives
$\frac{\partial^\lambda c_2}{\partial x^\lambda}$	=	fractional-in-Space derivatives with respect to the horizontal flow
$\frac{\partial^\lambda c_2}{\partial y^\lambda}$	=	fractional-in-Space derivatives with respect to the vertical flow

The initial and boundary conditions are given as;

$$\overline{D}_2\left(\frac{\partial^\lambda c_2}{\partial y^\lambda}\right) = D_1\left(\frac{\partial^\alpha c_1}{\partial y^\alpha}\right) \quad \text{at} \quad y=0 \tag{2.6}$$

$$\frac{\partial^\lambda c_2}{\partial y^\lambda} = 0 \qquad \text{at} \ \ y = -h \tag{2.7}$$

$$c = \frac{1}{h}\int_{-h}^{0} c_2\, dy \tag{2.8}$$

Method of Solution

Applying Duh amel's Theorem (Randall and Leveque, 2005), and for a uniform source of contamination at x=0, we obtain the solution of problem (2.5) as;

$$\varphi(t,x) = 1 - \left(\frac{-1+e^{-t}}{t} - \frac{-1+e^{Xb\gamma\cos(\pi\gamma)+X\pi\cos\beta^2}}{x(b\gamma\cos(\pi\gamma)+\pi\cos\beta^2)}\right)\left(\frac{\sin xb\sin\pi\gamma}{\gamma+1} - \frac{x\sin\pi\beta}{\beta+1}\right) \tag{2.9}$$

and

$$\frac{\partial}{\partial t}\int_0^t C_0(t-\tau)\varphi(\tau - x, x)\,d\tau \qquad (2.10)$$

ρ is defined by equation (2.9)

$$\left(\left(\frac{-1+e^{-t}}{t}\right) - \left(\frac{xb\gamma + xt^{\beta}\left(\frac{1}{2}\gamma\right)+\frac{1}{2}}{1-\gamma}\right) + \left(\frac{xt^{\gamma}\left(\frac{1}{2}\beta\right)+\frac{1}{2}}{1+\beta}\right)\right) \qquad (2.11)$$

;

$$\frac{\Gamma(-\beta-1)\Gamma(-\gamma+1)+xt^{\beta}\Gamma(-\gamma+1)+bxt^{\gamma}\Gamma(-\beta-1)}{\Gamma(-\beta-1)\Gamma(-\gamma+1)} \qquad (2.12)$$

$$\beta = \frac{\alpha}{\alpha+1} \qquad (2.13)$$

e;

$$\Gamma(\beta)=\int_0^1 t^{\beta-1}e^{-t}\,dt\,; \quad \text{and} \quad \Gamma(\beta+1)=\int_0^1 t^{\beta}e^{-t}\,dt\,; \qquad (2.14)$$

= Diffusive Transport into the unconfined Layer

= Diffusive Transport within the Layer

= Size of the Aquifer

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quifer Concentration Distribution

able 3.1: Concentration Distribution Values for $\alpha, b,$ and γ

α	b	γ	t
			$0 < t \le 7$
1.00	0.50	0.50	

$$(t,x)=\frac{\partial}{\partial t}\int_0^t C_0(t-\tau)\varphi(\tau-x,x)\,d\tau \qquad (2.10)$$

here;

φ is defined by equation (2.9)

nd

$$C=1-\left(\left(\frac{-1+e^{-t}}{t}\right)-\left(\frac{xb\gamma+xt^{\beta}\left(\frac{1}{2}\gamma\right)+\frac{1}{2}}{1-\gamma}\right)+\left(\frac{xt^{\gamma}\left(\frac{1}{2}\beta\right)+\frac{1}{2}}{1+\beta}\right)\right) \qquad (2.11)$$

hat is;

$$C=\frac{-\Gamma(-\beta-1)\Gamma(-\gamma+1)+xt^{\beta}\Gamma(-\gamma+1)+bxt^{\gamma}\Gamma(-\beta-1)}{\Gamma(-\beta-1)\Gamma(-\gamma+1)} \qquad (2.12)$$

and,

$$\beta=\frac{\alpha}{\alpha+1} \qquad (2.13)$$

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α = Diffusive Transport into the unconfined Layer

γ = Diffusive Transport within the Layer

b = Size of the Aquifer

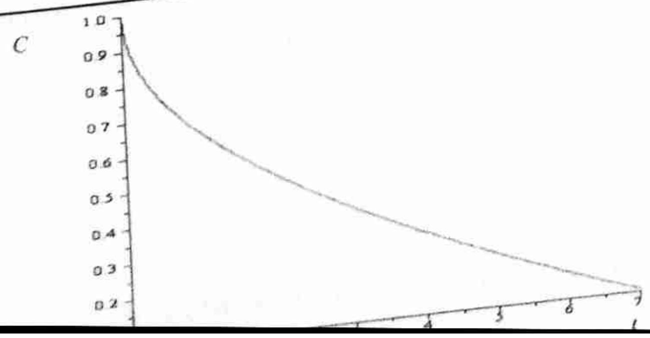
Solution (2.12) describes the behaviour of contaminants transport under a uniform source of contamination $C^U(x)$ for different values of values of α, b and γ for $0<t\le 7$ as shown in figures 3.1, 3.2 and 3.3 respectively.

Results

Aquifer Concentration Distribution

Table 3.1: Concentration Distribution Values for $\alpha, b,$ and γ

α	b	γ	t
1.00	0.50	0.50	$0<t\le 7$



$$C(t,x) = \frac{\partial}{\partial t}\int_0^t C_0(t-\tau)\varphi(\tau - x, x)\,d\tau \qquad (2.10)$$

where;

φ is defined by equation (2.9)

and

$$C = 1 - \left(\left(\frac{-1+e^{-t}}{t} \right) - \left(\frac{xb\gamma + xt^{\beta}\left(\frac{1}{2}\gamma\right) + \frac{1}{2}}{1-\gamma} \right) + \left(\frac{xt^{\gamma}\left(\frac{1}{2}\beta\right) + \frac{1}{2}}{1+\beta} \right) \right) \qquad (2.11)$$

That is;

$$C = \frac{-\Gamma(-\beta-1)\Gamma(-\gamma+1) + xt^{\beta}\Gamma(-\gamma+1) + bxt^{\gamma}\Gamma(-\beta-1)}{\Gamma(-\beta-1)\Gamma(-\gamma+1)} \qquad (2.12)$$

and,

$$\beta = \frac{\alpha}{\alpha+1} \qquad (2.13)$$

where;

$$\Gamma(\beta) = \int_0^1 t^{\beta-1}e^{-t}\,dt; \quad \text{and} \quad \Gamma(\beta+1) = \int_0^1 t^{\beta}e^{-t}\,dt; \qquad (2.14)$$

α = Diffusive Transport into the unconfined Layer

γ = Diffusive Transport within the Layer

b = Size of the Aquifer

Solution (2.12) describes the behaviour of contaminants transport under a uniform source of contamination $C^U(x)$ for different values of values of α, b and γ for $0 < t \le 7$ as shown in figures 3.1, 3.2 and 3.3 respectively.

Results

Aquifer Concentration Distribution

Table 3.1: Concentration Distribution Values for $\alpha, b,$ and γ

α	b	γ	t
1.00	0.50	0.50	$0 < t \le 7$

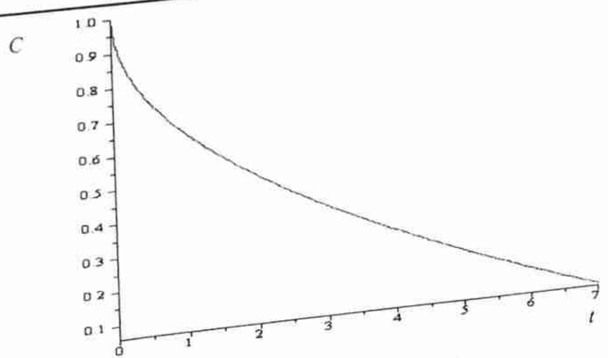


Figure 3.1: Aquifer Concentration Distribution for Values of $\alpha, b,$ and γ

Table 3.2: Concentration Distribution Values for $\alpha, b,$ and γ

α	b	γ	t
0.50	0.50	0.50	$0 < t \le 7$

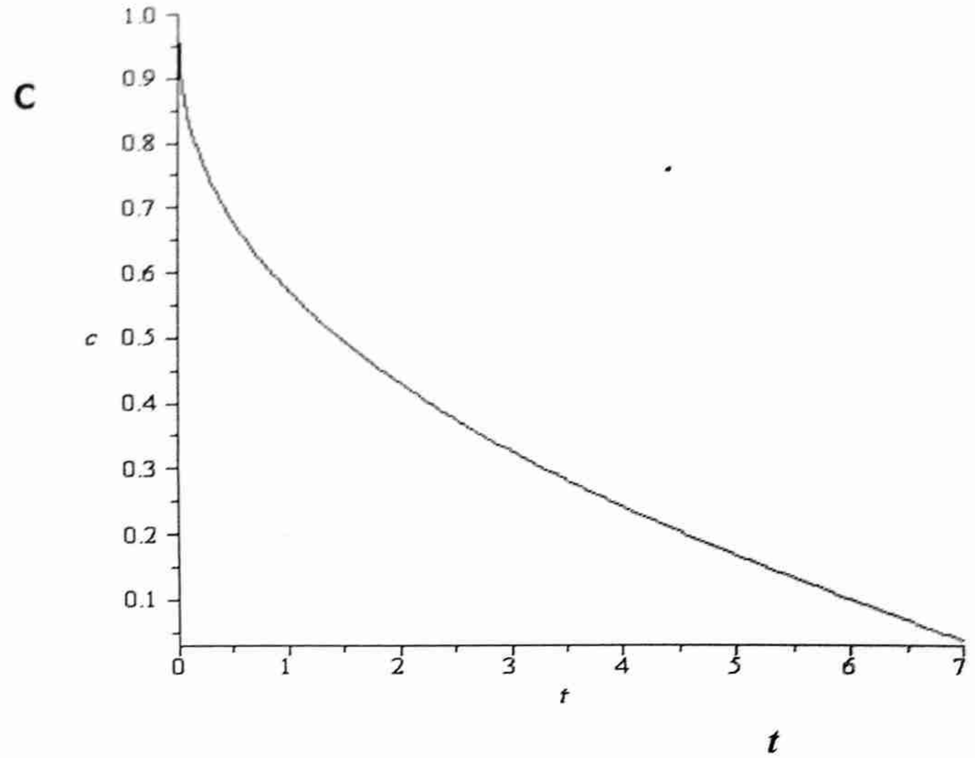


Figure 3.2: Aquifer Concentration Distribution for Values of $\alpha, b,$ and γ

Table 3.3: Concentration Distribution Values for $\alpha, b,$ and γ

α	b	γ	t
0.20	0.50	0.50	$0 < t \le 7$

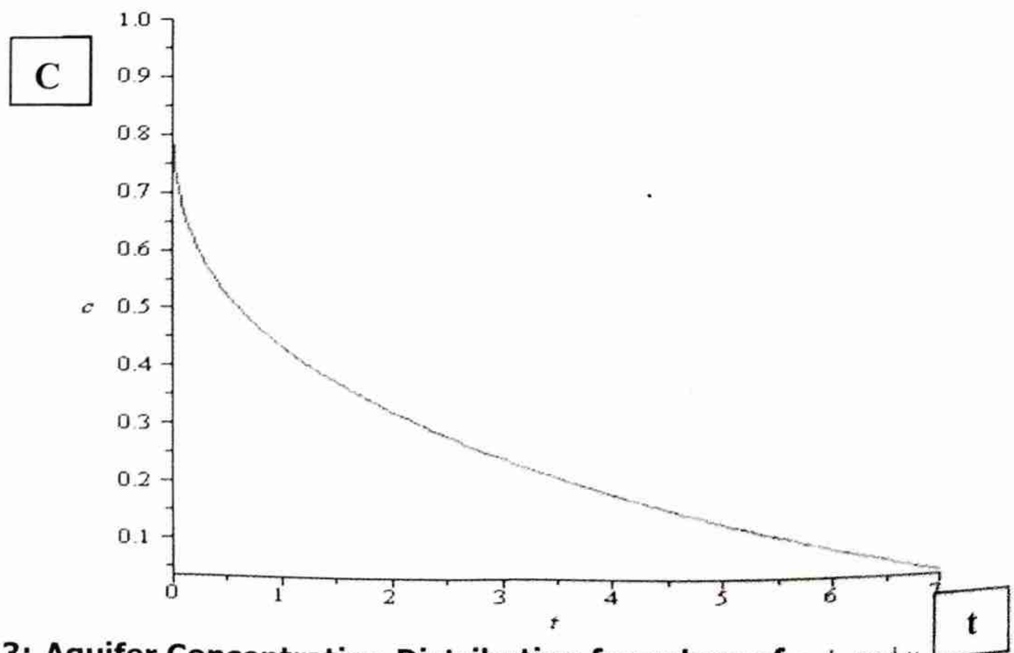


Figure 3.3: Aquifer Concentration Distribution for values of $\alpha, b,$ and γ

Table 3.4: Concentration Distribution for Different Values of $\alpha, b,$ and γ

α	b	γ	t
1.00	0.50	0.50	$0 < t \leq 7$
0.50	0.50	0.50	$0 < t \leq 7$
0.20	0.50	0.50	$0 < t \leq 7$

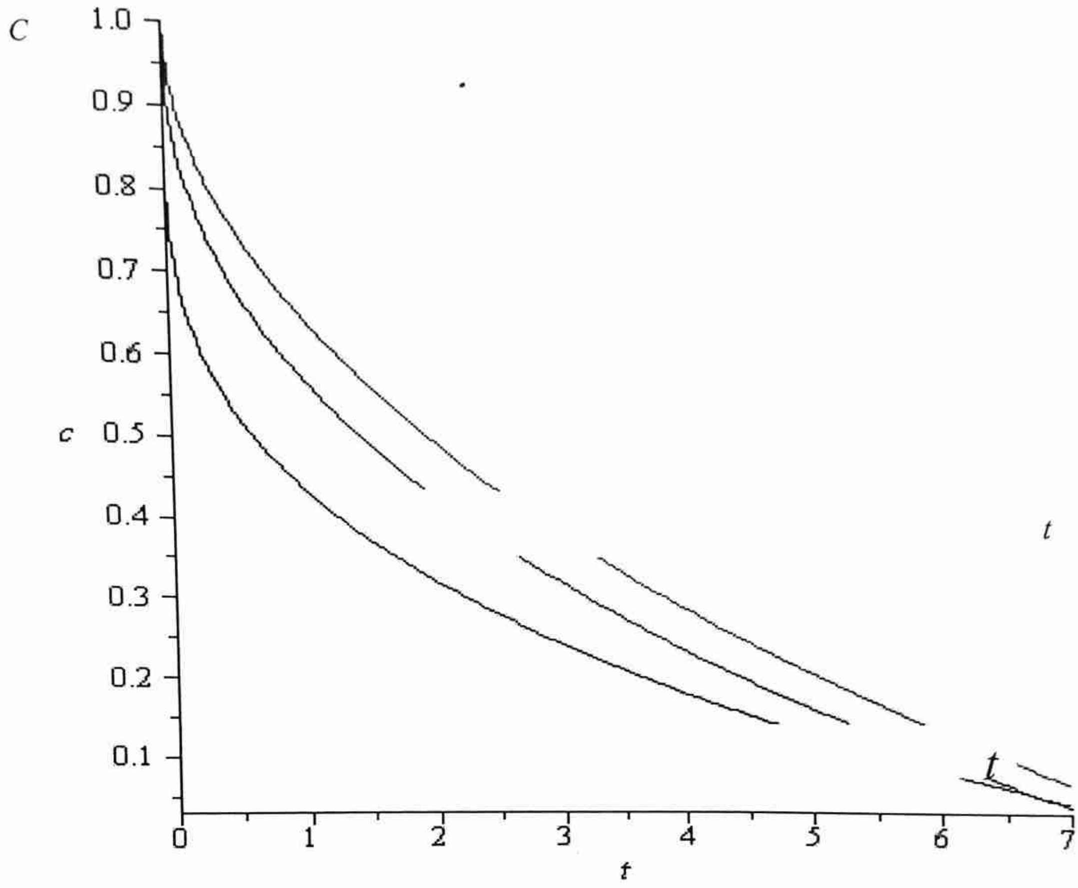


Figure 3.4: Aquifer Concentration Distribution for Different Values of $\alpha, b,$ and γ

Conclusion

The results indicate that for a uniform source of contamination, $C^U(x)$, for values of α = 1.00, b =0.5, and γ =0.5 the level of contamination is at its peak as shown in figure 3.1. A reduction of the value α to 0.50, we noticed a reduction in the level of concentration as shown in figure 3.2. A further reduction of the value of α to 0.20, we noticed a further reduction in the level of concentration as shown in figure 3.3. This shows that for a uniform source of contamination, the level of contamination reduces over time depending on the values of α , b and γ .

References

Amlan, D., & Bithin, D. (2015). Ground water management. *Journal of Sadhana*, 26(4), 87-99.

Fetter, C. W. (2014). Applied hydrogeology. New Delhi-India: *CBS Publishers & Distributors Pvt.* Ltd.

Helmut, D. (2014). Diffusion in solids. *Springer Berlin Heidelberg. New York*, 2(1), 56-58.

Jacques, A. (2014). The handbook of groundwater engineering. *CRC Press LLC 2000 Corporate Blvd.*, N.W. Boca Raton, FL 33431, U.S.A. 3(4), 89-100.

Kumar, C. P. (2014). Groundwater flow models. *National Institute of Hydrology*, Roorkee –247667. 2(3), 33-45

Olasehinde, P. I. (2014). The groundwaters of Nigeria: A solution to sustainable national water needs. *Inaugural Lecture Series 17*, Federal University of Technology, Minna, Nigeria.

Randall, R., & Leveque, G. (2005). Application of Lie group analysis in group transformation for fluid flow. *Superfund Technology Support Center for Ground Water*, EPA/540/S-92/005 U.S. 34-35.

Sagei, E., Vdimir, C., & Toiyuki, H. (2015). The effect of non-Fickian diffusion into surrounding rocks on contaminant transport. *Journal of Mathematical and Engineering Sciences*, 3(4), 12-25.

Thangarajan, A. (2014). Groundwater resource evaluation and augmentation. New Delhi, India: *Capital Publishing Company*, 3(5), 45 - 51.