APPLICATION OF ACCELERATED OVERRELAXATION METHOD TO THE NUMERICAL SOLUTION OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract
In this paper, the finite difference method is employed to discretize the partial differential equation (PDE) through replacement of the PDE by a difference equation to be satisfied by the values of the vector of unknowns x at a finite set of points in the domain of the independent variable. This discretisation ultimately results in an associated linear system of equations $Ax = b$, where A is an $n-square$ matrix, b is an $n \times 1$ column vector and x is the vector of unknowns. A large body of iterative methods for solving such linear systems abound, and several of them have been studied in order to improve on their robustness, convergence and suitability for specialized systems. One of such methods is the Accelerated Overrelaxation (AOR). The AOR is a two-parameter generalization of the classical Jacobi, Gauss-Seidel and Successive Overrelaxation (SOR) methods for the iterative solution of the linear system $Ax = b$. Here, the basics of the AOR method is established, suitable values are assigned to the parameters involved, and the method is applied to solve some partial differential equations of elliptic type. Results of numerical experiments proved the effectiveness of the method.

Keywords: *Accelerated Overrelaxation Method, Iterative Method, Elliptic Partial Differential Equation, $L-matrix$, Spectral Radius*

Introduction

The numerical solution of partial differential equations (PDEs) more often than not involves discretization; this entails the approximation of the PDEs by equations that involve a finite number of unknowns. The finite difference method (FDM) is one simple method of discretizing a PDE; it involves replacement of the PDE by a difference equation which must be satisfied by the values of the unknown function x at a finite set of points in the domain Ω(Saad, 2000).Using this method, the domain is divided into a finite number of nodes or meshpoints, where each node is assigned a unique identifier based on its position in the mesh. The approximation of partial derivatives by finite differences more often than not leads to an associated linear system of equations

$$Ax = b \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a nonsingular matrix with nonvanishing diagonal elements, and where $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ are respectively vectors of unknown and preassigned variables. We also consider the usual splitting of $A = D - L_A - U_A$ such that

$$D_{ij} = \begin{cases} A_{ij} \text{if } i = j, \\ 0 \ \text{ otherwise}, \end{cases} \quad -L_{A_{ij}} = \begin{cases} A_{ij} \text{if } j < i, \\ 0 \ \text{ otherwise}, \end{cases} \quad -U_{A_{ij}} = \begin{cases} A_{ij} \text{if } j > i, \\ 0 \ \text{ otherwise}. \end{cases}$$

For simplicity, we impose the normalization $D^{-1}Ax = D^{-1}b$ on (1) such that $A = I - L - U$, where I is the identity matrix, and $-L$ and $-U$ are respectively the strictly lower and strictly upper triangular matrices.
To approximate the solution of the linear system (1) we can use the Accelerated Overrelaxation (AOR) iterative method introduced by Hadjidimos (1978). The iteration matrix of the AOR method, denoted by $L_{r,\omega}$, is defined by

$$L_{r,\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U] \tag{2}$$

where r and $\omega(\neq 0)$ are scalars called acceleration and overrelaxation parameters respectively. It is well-known that for specific values of the parameters r and ω, the following standard methods can be obtained as special cases of the AOR: Jacobi ($L_{0,1}$), Gauss-Seidel ($L_{1,1}$), Jacobi overrelaxation (JOR) ($L_{0,\omega}$) and successive overrelaxation (SOR) ($L_{\omega,\omega}$). Therefore, the AOR method is a two-parameter generalization of the most popular basic iterative methods. A basic iterative method is a one-step method of the form $x^{(n)} = Gx^{(n-1)} + k$ where for some nonsingular matrix Q we have $G = I - Q^{-1}A$ and $k = Q^{-1}b$. But the AOR method, except for the case $r = 0$, is essentially a one-parameter extrapolation of the SOR (ESOR) method with overrelaxation r and extrapolation ω(Hadjidimos, 1978).

$$L_{r,\omega} = sL_{r,r} + (1-s)I \tag{3}$$

With specific conditions imposed on the coefficient matrix A, and some restrictions on the parameters r and ω, Hadjidimos (1978) established convergence of AOR for three cases: irreducible matrices with weak diagonal dominance, $L-$ matrices and consistently ordered matrices.In Hallett (1986)convergence of AOR was extended to cover any real-valued equation system. It is known that the AOR is a fast converging method in relation to

SOR; however, its convergence rate can be improved so as to appeal to its applicability in industry. In this light, several researchers, Evans *et al.* (2001), Li *et al.* (2007), Nasabzadeh and Toutounian (2013), Ren*et al.* (2016),Salkuyeh and Abdolalizadeh (2011), Wu and Liu (2014), Wu *et al.* (2007) and Youssef and Farid(2015)have made significant contributions.
This present work is an attempt to employ the AOR iterative method to approximate the solution of sparse linear systems arising from the discretisation of self-adjoint partial differential equations of elliptic type.

Materials And Methods
Let us consider the linear self-adjoint elliptic partial differential equation

$$\nabla^2 u = -1 \quad (4)$$

$$u = 0, \qquad |x| = 1, \qquad |y| = 1 \quad (5)$$

defined in the unit square, $|x| \le 1, |y| \le 1$.
To solve (4), a square mesh of vertical and horizontal lines with mesh spacing $h = 1/4$ in both the vertical and horizontal directions is super imposed over the square region $0 \le x \le 1,\ 0 \le y \le 1$. This results in 9internal pointsand 16 boundary points as shown in Figure 1.

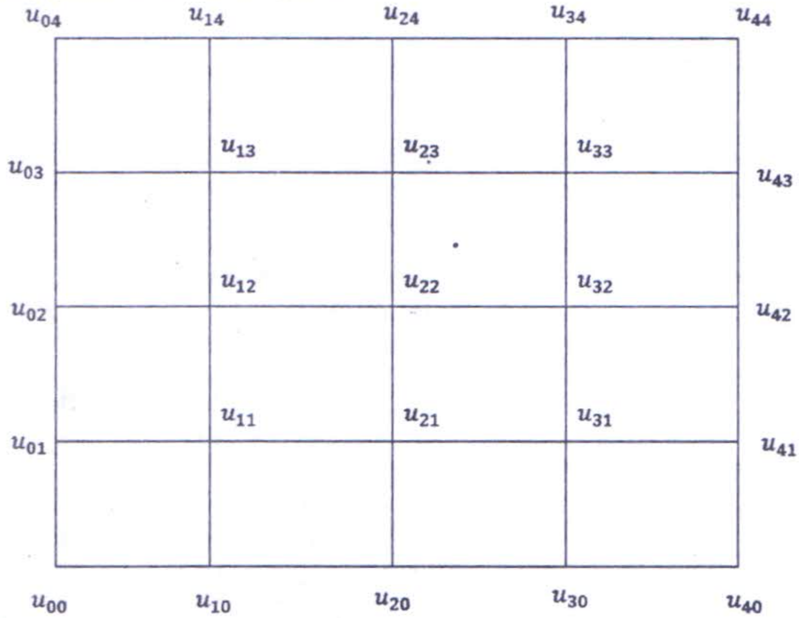

Figure 1 Discretisation of square region $|x| \le 1, |y| \le 1$with 4×4density
At each interior point of the region, the partial derivatives$\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ appearing in equation (4) are replaced by the standard second order three-point central difference quotients$(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2$ and $(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})/h^2$ respectively. Thus the finite difference approximation to equation (4) at each interior grid point is given by

$$(x+1)[(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2] + (y^2+1)[(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})/h^2] + u_{i,j} = -1 \quad (6)$$

i.e.,

$$[2(x + y^2 + 2) - h^2]u_{i,j} - (x+1)u_{i+1,j} - (x+1)u_{i-1,j} - (y^2+1)u_{i,j+1} - (y^2+1)u_{i,j-1} = h^2 (7)$$

At the interior points, $u_{11}, u_{21}, u_{31}, u_{12}, u_{22}, u_{32}$, u_{13}, u_{23} and u_{33}, the values of (x,y) are, respectively,$(-1/2\,,-1/2)$, $(0,-1/2)$, $(1/2\,,-1/2)$,$(-1/2\,,0)$, $(0{,}0)$, $(1/2\,,0)$, $(-1/2\,,1/2)$, $(0,1/2)$ and $(1/2\,,1/2)$.
The known boundary values are $u_{01} = 0, u_{01} = 0, u_{20} = 0, u_{41} = 0, u_{30} = 0, u_{02} = 0, u_{42} = 0,\ u_{03} = 0$, $u_{14} = 0,\ u_{24} = 0,\ u_{43} = 0, u_{34} = 0$.
Equation (7) is applied at each interior point to obtain the following linear system

$$\left.\begin{aligned}
4u_{11} - u_{21} - u_{01} - u_{12} - u_{10} &= \frac{1}{16} \\
4u_{21} - u_{31} - u_{11} - u_{22} - u_{20} &= \frac{1}{16} \\
4u_{31} - u_{41} - u_{21} - u_{32} - u_{30} &= \frac{1}{16} \\
4u_{12} - u_{22} - u_{02} - u_{13} - u_{11} &= \frac{1}{16} \\
4u_{22} - u_{32} - u_{12} - u_{23} - u_{21} &= \frac{1}{16} \\
4u_{32} - u_{42} - u_{22} - u_{33} - u_{31} &= \frac{1}{16} \\
4u_{13} - u_{23} - u_{03} - u_{14} - u_{12} &= \frac{1}{16} \\
4u_{23} - u_{33} - u_{13} - u_{24} - u_{22} &= \frac{1}{16} \\
4u_{33} - u_{43} - u_{23} - u_{34} - u_{32} &= \frac{1}{16}
\end{aligned}\right\} \qquad (8)$$

The known boundary values are further substituted into (8) to obtain the matrix vector notation $Ax = b$

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = \begin{pmatrix} 1/16 \\ 1/16 \\ 1/16 \\ 1/16 \\ 1/16 \\ 1/16 \\ 1/16 \\ 1/16 \\ 1/16 \end{pmatrix} \qquad (9)$$

Then the linear system (9) is expressed in matrix vector notation $D^{-1}Ax = D^{-1}b$, for the AOR method (2) to be applied. The spectral radius of the AOR method is computed for various values of acceleration and overrelaxation parameters r and ω and the results are compared with those of the SOR method.

Results and Discussion of Findings

Table 1 Spectral radii of iteration matrices for various values of r and ω.

ω	r	$\rho(L_{r,\omega})$	$\rho(L_{sor})$
0.1	0.2	0.9684428864	0.9696286079
0.2	0.3	0.9342686035	0.9368857754
0.3	0.4	0.8970329252	0.9014029066
0.4	0.5	0.8561552823	0.8627105745
0.5	0.6	0.8108495299	0.8201941016
0.6	0.7	0.7600000002	0.7730194340
0.7	0.8	0.7019160160	0.7200000000
0.8	0.9	0.6337715516	0.6593325909
0.9	0.95	0.5705360266	0.5879929947

In Table 1, we denote $L_{r,\omega}$ and L_{sor} as iteration matrices of AOR and SOR methods respectively; the parameters r and ω are respectively the acceleration and overrelaxation parameters. Table 1 illustrates that convergence of the AOR iterative method is faster than that of the SOR iterative method.

Conclusion

In this paper we have demonstrated the importance of the AOR (SOR) method as a very simple and powerful technique for solving linear systems arising from finite difference discretisation of elliptic partial differential equations. It should come as a no surprise that the AOR method exhibits faster convergence than the SOR; this is due to the presence of two parameters as opposed to the one parameter of SOR. Further research effort could be devoted to finding the optimum combination of the two parameters, for faster convergence.

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