A02: The Stability Analysis of a Block Hybrid Implicit Runge-Kutta Type Method for an Initial Value Problem

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Abstract

In this research paper, we present the convergence and stability analysis of an implicit Runge-Kutta Type Method (RKTM) for direct integration of first and second order ordinary differential equation (ODE). In the process, we obtained the order and error constant, test for consistency and convergence and plot the region of absolute stability.

Keywords: Convergence, Implicit, Runge-Kutta Type, Stability,

1. Introduction

One of the basic properties of an acceptable numerical method is that the solution generated by the method converges and stable in some sense to the exact solution as the steplength h tends to zero. The stability properties of a numerical method help us to determine a good approximation to the exact solution.

The initial value problem (IVP) for first order Ordinary Differential Equation is defined by

$$y' = f(x, y) \qquad y(x_0) = y \qquad x \in [a, b] \tag{1}$$

The general s-stage Runge-Kutta method is defined by

$$y_{n+1} = y_n + h\sum_{i,j=1}^{s} a_{ij}k_i \tag{2a}$$

where for $i = 1,2 \ldots\ldots\ldots s$

$$k_i = f(x_i + c_j h, y_n + h\sum_{i,j=1}^{s} a_{ij}k_j) \tag{2b}$$

The real parameters c_j, k_i, a_{ij} define the method. The method in Butcher array form can be expressed as

$$\begin{array}{c|c} c & \beta \\ \hline & b^T \end{array}$$

Where $a_{ij} = \beta$

Definition 1: Consistency of Runge-Kutta Methods

The first and second order Ordinary Differential Equation (ODE) methods are said to be consistent if

$$\varphi(x, y(x), 0) \equiv f(x, y(x)) \qquad (3)$$

$$\varphi(x, y(x), y'(x), 0) \equiv f(x, y(x), y'(x)) \qquad (4)$$

holds respectively.

Note that consistency demands that $\sum_1^s b_s = 1$, and $\sum_1^s b_s = \frac{1}{2}$ for first and second order respectively. Also $\sum_1^s b_s$ is as shown in the Butcher array table.

α	$\bar{A}$	A
	$\bar{b}^T$	b

$A = a_{ij} = \beta^2 \qquad \bar{A} = \bar{a}_{ij} = \beta \qquad \beta = \beta e$

Definition 2: Convergence of Runge –Kutta Methods

If $y' = f\big(x, y(x)\big)$; $\; y'' = f\big(x, y(x), y'(x)\big)$ represents first and second order respectively, then for such method consistency is necessary and sufficient for convergence. Hence the methods are said to be convergent if and only if they are consistent (Adegboye, 2013).

3. Methodology

To determine the order and error constant

For $c_1, c_2, \ldots., c_s$ and $k_1, k_2, \ldots\ldots k_s$ in (2b) we shall let $k_i = f_{c_i}$ implies $k_1 = f_{c_1}$, $k_2 = f_{c_2}$, $k_3 = f_{c_3}$ and $k_s = f_{c_s}$.

The Block Hybrid RKTM when $K = 1$ is given as

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= y_n + h(\tfrac{3}{4}k_2 - \tfrac{1}{4}k_3) \\ y_{n+1} &= y_n + hk_2 \end{aligned}\right\} \qquad (5)$$

Where

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \tfrac{1}{2}h, y_n + h\{0k_1 + \tfrac{3}{4}k_2 - \tfrac{1}{4}k_3\}) \\ k_3 &= f(x_n + h, y_n + h\{0k_1 + k_2 + 0k_3\}) \end{aligned}\right\} \quad (6)$$

Since $k_i = f_{c_i}$, implies $k_1 = f_{c_1}, k_2 = f_{c_2}, k_3 = f_{c_3}$ (7)

From equation (2b)

$$c_1 = 0,\ c_2 = \frac{1}{2}, c_3 = 1\ . \quad (8)$$

Therefore $k_1 = f_0 = f_n, k_2 = f_{\frac{1}{2}} = f_{n+\frac{1}{2}}, k_3 = f_1 = f_{n+1}$ (9)

Then equation (5) now becomes

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= y_n + h(\tfrac{3}{4}f_{n+\frac{1}{2}} - \tfrac{1}{4}f_{n+1}) \\ y_{n+1} &= y_n + hf_{n+\frac{1}{2}} \end{aligned}\right\} \quad (10)$$

Taylor series expansion of

$$y\left(n+\frac{1}{2}h\right) = y(n) + \frac{1}{2}hy'(n) + \frac{\left(\frac{1}{2}h\right)^2}{2!}y''(n) + \frac{\left(\frac{1}{2}h\right)^3}{3!}y'''(n) + \cdots + \frac{\left(\frac{1}{2}h\right)^s}{s!}y^s(n) \quad (11)$$

$$y(n+h) = y(n) + hy'(n) + \frac{(h)^2}{2!}y''(n) + \frac{(h)^3}{3!}y'''(n)\ \ldots + \frac{\left(\frac{1}{2}h\right)^s}{s!}y^s(n) \quad (12)$$

$$f\left(n+\frac{1}{2}h\right) = f_{\frac{1}{2}} = y'(n) + \frac{1}{2}hy''(n) + \frac{\left(\frac{1}{2}h\right)^2}{2!}y'''(n) + \cdots + \frac{\left(\frac{1}{2}h\right)^{(s-1)}}{(s-1)!}y^s(n) \quad (13)$$

$$f(n+h) = f_1 = y'(n) + hy''(n) + \frac{(h)^2}{2!}y'''(n) + \frac{(h)^3}{3!}y'^v(n) + \cdots + \frac{(h)^{(s-1)}}{(s-1)!}y^s(n) \quad (14)$$

By substituting the taylor series expansion into equation (3.68), we have

$$y_{n+\frac{1}{2}} - y_n - h\left(\frac{3}{4}f_{n+\frac{1}{2}} - \frac{1}{4}f_{n+1}\right) = \frac{5}{96}h^3y^3 \quad (15)$$

The method is of order 2 and the error constant is $\frac{5}{96}$.

Also,

$$y_{n+1} - y_n - hf_{n+\frac{1}{2}} = \frac{1}{24}h^3y^3 \quad (16)$$

The method is of order 2 and the error constant is $\frac{1}{24}$

The second derivative when $\boldsymbol{k = 1}$ is given as

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= y_n + hy'_n + h^2\left(0k_1 + \frac{5}{16}k_2 - \frac{3}{16}k_3\right), \\ y'_{n+\frac{1}{2}} &= y'_n + h\left(0k_1 + \frac{3}{4}k_2 - \frac{1}{4}k_3\right) \\ y_{n+1} &= y_n + hy'_n + h^2\left(0k_1 + \frac{3}{4}k_2 - \frac{1}{4}k_3\right), \\ y'_{n+1} &= y'_n + h(0k_1 + k_2 + 0k_3) \end{aligned}\right\} \qquad (17)$$

From equation (2b)

$$c_1 = 0,\ c_2 = \frac{1}{2}, c_3 = 1\ . \qquad (18)$$

Therefore

$$k_1 = f_0 = f_n, k_2 = f_{\frac{1}{2}} = f_{n+\frac{1}{2}}, k_3 = f_1 = f_{n+1} \qquad (19)$$

the equation now becomes

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= y_n + hy'_n + h^2\left(0f_n + \frac{5}{16}f_{n+\frac{1}{2}} - \frac{3}{16}f_{n+1}\right), \\ y'_{n+\frac{1}{2}} &= y'_n + h\left(0f_n + \frac{3}{4}f_{n+\frac{1}{2}} - \frac{1}{4}f_{n+1}\right) \\ y_{n+1} &= y_n + hy'_n + h^2\left(0f_n + \frac{3}{4}f_{n+\frac{1}{2}} - \frac{1}{4}f_{n+1}\right), \\ y'_{n+1} &= y'_n + h\left(0f_n + f_{n+\frac{1}{2}} + 0f_{n+1}\right) \end{aligned}\right\} \qquad (20)$$

The taylor series expansion of

$$f_{\frac{1}{2}} = f\left(n + \frac{1}{2}h\right) = y'' + \left(\frac{1}{2}h\right)y''' + \frac{(\frac{1}{2}h)^2}{2!}y^{iv} + \cdots \ldots\ldots + \frac{(\frac{1}{2}h)^s}{s!}y^{s+2} \qquad (21)$$

$$f_1 = f(n + h) = y'' + hy''' + \frac{(h)^2}{2!}y^{iv} + \ldots\ldots\ldots\ldots + \frac{(h)^s}{s!}y^{s+2} \qquad (22)$$

3.1 Figures and Tables

This is the table for the order and error constant for second derivative of k = 1block hybrid RKTM

Table 1: The Order and Error Constant for the second derivative for K=1 block hybrid RKTM.

Method	Order	Error Constant
$y_{n+\frac{1}{2}} = y_n + hy_n' + h^2\left(0k_1 + \frac{5}{16}k_2 - \frac{3}{16}k_3\right)$	2	$\frac{5}{96}$
$y_{n+1} = y_n + hy_n' + h^2\left(0k_1 + \frac{3}{4}k_2 - \frac{1}{4}k_3\right)$	2	$\frac{1}{24}$

4. Results and Discussion

From definition 1 and 2, that states the conditions for consistency and convergence of Runge-Kutta methods

The methods (5) for the first derivative $K = 1$ as indicated in the Butcher Table

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{3}{4} & \frac{-1}{4} \\ 1 & 0 & 1 & 0 \\ \hline & 0 & 1 & 0 \end{array}$$

are consistent since $\sum_1^s b_s = 1$, hence convergent.

Also the method (17) for the second derivative of $K = 1$ are consistent

$$\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{5}{16} & \frac{-3}{16} \\ 1 & 0 & 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{4} \\ \hline & 0 & 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{4} \end{array}$$

since $\sum_1^s b_s = \frac{1}{2}$, hence convergent.

The matrices for stability polynomial M $=\begin{bmatrix} A & U \\ B & V \end{bmatrix}$ are given below

Where $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{-1}{4} \\ 0 & 1 & 0 \end{bmatrix}$ $\quad U = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad B = [0 \quad 1 \quad 0]$ $\quad V = [1]$

The characteristic polynomial and stability function are as follows respectively:

$$M(z) = V + zB(1 - zA)^{-1}\mathrm{U} \quad (23)$$

$$= \frac{-z-4-3\eta z+4\eta+\eta^{\ 2}}{z^2-3z+4} \quad (24)$$

$$\phi(\eta, z) = \det\big(\eta I - m(z)\big) \quad (23)$$

$$= \frac{z^2+8z-1}{(z^2-3z+4)^2} \quad (24)$$

Putting the characteristic polynomial and stability function in a MATLAB software shows the region of absolute stability to be A stable.

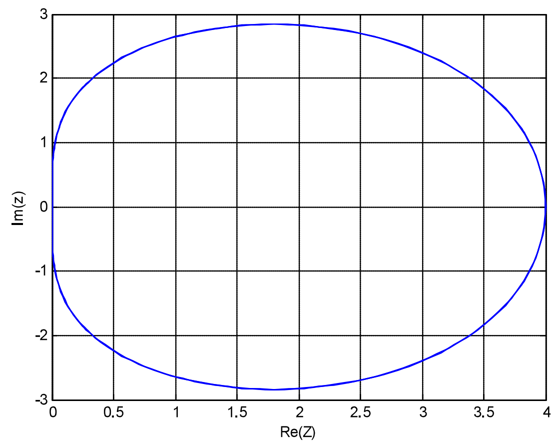

Figure 1: Stability region for $K = 1$ first and second order RKTM

5. Conclusion

For the Block Hybrid Runge-Kutta Type Method (BHRKTM) with step number $k = 1$, each of the stages reformulated into linear multistep method and with the aid of Taylor series expansion gave rise to the uniform order with their error constant. All the methods that formed the block are of uniform order 2 with varying error constants. The procedures highlighted step by step and results obtained tabulated helped to establish the consistency and convergence of the methods. This helps to determine a standard method to adopt at any point in time. The procedure adopted explained a simple approach that speeds up computation and reduces computational effort in determining the order, error constant, consistency and convergence a Runge- Kutta Type Method (RKTM). This will also serve as a guide for researchers on how to determine the order, error constant and convergence of a Block Hybrid Runge-Kutta Type Method (BHRKTM). It will also help to determine a good choice of the method.

References

Adegboye, Z.A (2013). Construction and implementation of some reformulated block implicit linear multistep method into runge-kutta type method for initial value problems of general second and third order ordinary differential equations. Unpublished doctoral dissertation, Nigerian Defence Academy, Kaduna .

Agam, A.S (2013). A sixth order multiply implicit Runge-kutta method for the solution of first and second order ordinary differential equations. Unpublished doctoral dissertation, Nigerian Defence Academy, Kaduna .

Butcher, J.C (2005). “Runge-Kutta method for ordinary differential equations” COE workshop on numerical analysis Kyushu university pp1-208

Muhammad R , Yahaya Y.A, & Abdulkareem A.S. (2015).Error and Convergence Analysis of a Hybrid Runge - kutta Type Method. *International Journal of Science and Technology (IJST,)* 4(4) 164-168.

Tamer A.Abassy (2000). Introduction to Piecewise Analytic Method. *Journal of Practitional Calculus and Application* 3(8) 1-19.

Yahaya, Y.A. & Adegboye, Z.A. (2011). Reformulation of quade’s type four-step block hybrid multstep method into runge-kutta method for solution of first and second order ordinary differential equations. *Abacus, 38(2)*, 114-124.

Yahaya Y.A. and Adegboye Z.A. (2013). Derivation of an implicit six stage block runge kutta type method for direct integration of boundary value problems in second order ode using the quade type multistep method. *Abacus*, *40(2)*, 123-132.

Oyedum, O.D and Gambo, G.K. (1994). Surface Radio Refractivity in Northern Nigeria, *Nigerian Journal of Physics*, Vol. 6, 36-41.