A07: Construction of Block Hybrid Backward Differentiation Formula for some Classes of Second Order Initial Value Problems

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ABSTRACT

This paper focuses on the derivation of a 2-step block hybrid backward differentiation formula for the general solution of second order ordinary differential equations. The new method was derived using the procedure of collocation and interpolation of power series at some selected grid and off-grid points. Basic properties of the method were examined and the method found tobe zero stable, consistent and convergent. The method was tested on some classes of second orderinitial value problems and on comparing with the exact solutions, the method performed accurately with relatively small errors.

Keywords: Block Hybrid Backward Differentiation formula, second order IVPs, collocation and interpolation, convergence.

1.0 Introduction

Numerical analysis has always been used in different fields such as in Sciences, Medicines, Engineering, and in diverse facet of life. In recent times, the integration of Ordinary Differential Equations (ODEs) are investigated using some kind of block methods. Development of Linear Multi-step Method (LMM) for solving ODE can be generated using methods such as Taylor's series, numerical interpolation, numerical integration and collocation method, which are restricted by an assumed order of convergence. Milne (1953) proposed blockmethods for solving ODEs. The Milne's idea of proceeding in blocks was developed by (Rosser, 1967). The approach in providing solution to second order differential equations, is to first convert the problem to a system of first order ODEs and then solve using numerical method such as the Runge-Kutta method and liner multistep methods (Lambert, 1973 and 1991).

The studies on direct method for higher order ODEs reveal the advantages in speed and accuracy. The objective of numerical analysis is to solve complex numerical problems like stiff equation using only the simple operations of arithmetic, to develop and evaluate methods for computing numerical results from given data.Jator, (2010) and Mehrknoon(2011) investigates the complicated computational work and lengthy finishing time of numerical simulations. The backward differentiation formula (BDF) is a component of the family of implicit linear multistep methods for the numerical integration of ordinary differential equations. They are characterized by a single function evaluation point and region of absolute stability. They are linear multistep method that, for a given function and point, approximate the derivative of that function using information from already computed points, thereby increasing the accuracy of the approximation. These methods can be recommended for the general solution of second order ordinary differential equations.

2.0. Derivation of the Method

The derivation of Block Hybrid Backward Differentiation Formula (BHBDF) for solving some classes of second-order ordinary differential equations of the form

$$\frac{d^2y(x)}{dx^2} = f\left(x, y, \frac{dy(x)}{dx}\right) \tag{1}$$

coupled with appropriate initial conditions

$$y(x_0) = \varphi_1, \quad \frac{dy(x_0)}{dx} = \varphi_2 \tag{2}$$

is presented. Where f is a continuous function such that $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$, x_0 is the initial point, $y \in \mathbb{R}$ is an $n-$dimensional vector, x is a scalar variable, φ_1 and φ_2 are the initial values. The proposed BHBDF is of the form

$$Y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_v(x) y_{n+v} + h^2 \beta_k(x) f_{n+k} \tag{3}$$

where h is the chosen step size and $\alpha_j(x): j = 0,1,2,\ldots,k,\ \alpha_v(x), \beta_k(x)$ are unknown continuous coefficients to be determined. For Backward Differentiation Formula, we note that $\alpha_k = 1$ and $\beta_k \neq 0$. In this paper, a 2-step BHBDF (2BHBDF) method using power series function as the basis function is derived.

$$Y(x) = \sum_{j=0}^{t+c-1} p_j x^j \tag{4}$$

where t is the interpolation points, c is the collocation points and p_j are unknown coefficients to be determined. Then, taking

$$Y(x) = y_{n+j}, j = 0,1,2,\ldots,k-1 \tag{5}$$

$$Y''(x_{n+k}) = f_{n+k} \tag{6}$$

Interpolating (5) at $x_{n+i}; i = 0, \frac{1}{2}, 1, \frac{3}{2}$ and collocating (6) at $x_{n+i}; i = 2$. This results in a system of equations;

$$\psi X = Y \tag{7}$$

where

$\psi = \left(\alpha_0, \alpha_{\frac{1}{2}}, \alpha_1, \alpha_{\frac{3}{2}}, \beta_2\right)^T, Y = \left(y_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, f_{n+2}\right)^T$ and

$$X = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} \\ 1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} & x^4_{n+1} \\ 1 & x_{n+\frac{3}{2}} & x^2_{n+\frac{3}{2}} & x^3_{n+\frac{3}{2}} & x^4_{n+\frac{3}{2}} \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2} \end{pmatrix}$$

Solving (7) using matrix inversion approach with the aid of Maple 2015 software package to obtain the values of the unknown coefficients as;

$$\left.\begin{aligned}
\alpha_0 &= \frac{16}{35}\frac{x^4}{h^4} - \frac{284}{105}\frac{x^3}{h^3} + \frac{184}{35}\frac{x^2}{h^2} - \frac{421}{105}\frac{x}{h} + 1 \\
\alpha_{\frac{1}{2}} &= -\frac{8}{5}\frac{x^4}{h^4} + \frac{44}{5}\frac{x^3}{h^3} - \frac{72}{5}\frac{x^2}{h^2} + \frac{36}{5}\frac{x}{h} \\
\alpha_1 &= \frac{64}{35}\frac{x^4}{h^4} - \frac{332}{35}\frac{x^3}{h^3} + \frac{456}{35}\frac{x^2}{h^2} - \frac{153}{35}\frac{x}{h} \\
\alpha_{\frac{3}{2}} &= -\frac{24}{35}\frac{x^4}{h^4} + \frac{356}{105}\frac{x^3}{h^3} - \frac{136}{35}\frac{x^2}{h^2} + \frac{124}{105}\frac{x}{h} \\
\beta_2 &= \frac{2}{35}\frac{x^4}{h^2} - \frac{6}{35}\frac{x^3}{h} + \frac{11}{70}x^2 - \frac{3}{70}xh
\end{aligned}\right\} \tag{8}$$

The determined coefficients are then substituted into the continuous method in (3) to have

$$\begin{aligned}
y(x) = {} & \left(\frac{16}{35}\frac{x^4}{h^4} - \frac{284}{105}\frac{x^3}{h^3} + \frac{184}{35}\frac{x^2}{h^2} - \frac{421}{105}\frac{x}{h} + 1\right) y_n + \left(-\frac{8}{5}\frac{x^4}{h^4} + \frac{44}{5}\frac{x^3}{h^3} - \frac{72}{5}\frac{x^2}{h^2} + \frac{36}{5}\frac{x}{h}\right) y_{n+\frac{1}{2}} \\
& + \left(\frac{64}{35}\frac{x^4}{h^4} - \frac{332}{35}\frac{x^3}{h^3} + \frac{456}{35}\frac{x^2}{h^2} - \frac{153}{35}\frac{x}{h}\right) y_{n+1} \\
& + \left(-\frac{24}{35}\frac{x^4}{h^4} + \frac{356}{105}\frac{x^3}{h^3} - \frac{136}{35}\frac{x^2}{h^2} + \frac{124}{105}\frac{x}{h}\right) y_{n+\frac{3}{2}} \\
& + \left(\frac{2}{35}\frac{x^4}{h^2} - \frac{6}{35}\frac{x^3}{h} + \frac{11}{70}x^2 - \frac{3}{70}xh\right) f_{n+2}
\end{aligned} \tag{9}$$

Evaluating (3.10) at $x = x_{n+2}$, gives the discrete scheme as

$$y_{n+2} = -\frac{11}{35}y_n + \frac{8}{5}y_{n+\frac{1}{2}} - \frac{114}{35}y_{n+1} + \frac{104}{35}y_{n+\frac{3}{2}} + \frac{3}{35}h^2 f_{n+2} \tag{10}$$

To obtain the sufficient schemes required, the first derivative of (9) is obtained and evaluated at $x = x_n, x = x_{n+\frac{1}{2}}, x = x_{n+1}, x = x_{n+\frac{3}{2}}, x = x_{n+2}$ to obtain;

$$\left.\begin{aligned} hz_n &= -\frac{421}{105}y_n + \frac{36}{5}y_{n+\frac{1}{2}} - \frac{153}{35}y_{n+1} + \frac{124}{105}y_{n+\frac{3}{2}} - \frac{3}{70}h^2 f_{n+2} \\ hz_{n+\frac{1}{2}} &= -\frac{58}{105}y_n - \frac{7}{5}y_{n+\frac{1}{2}} + \frac{86}{35}y_{n+1} - \frac{53}{105}y_{n+\frac{3}{2}} + \frac{1}{70}h^2 f_{n+2} \\ hz_{n+1} &= \frac{23}{105}y_n - \frac{8}{5}y_{n+\frac{1}{2}} + \frac{19}{35}y_{n+1} + \frac{88}{105}y_{n+\frac{3}{2}} - \frac{1}{70}h^2 f_{n+2} \\ hz_{n+\frac{3}{2}} &= -\frac{34}{105}y_n + \frac{9}{5}y_{n+\frac{1}{2}} - \frac{162}{35}y_{n+1} + \frac{331}{105}y_{n+\frac{3}{2}} + \frac{3}{70}h^2 f_{n+2} \\ hz_{n+2} &= -\frac{17}{21}y_n + 4y_{n+\frac{1}{2}} - \frac{53}{7}y_{n+1} + \frac{92}{21}y_{n+\frac{3}{2}} + \frac{5}{14}h^2 f_{n+2} \end{aligned}\right\} \quad (11)$$

where z is the first derivative of y.

Likewise, we further obtain the second derivatives of (9), thereafter, evaluating at $x = x_{n+\frac{1}{2}}, x = x_{n+\frac{3}{2}}$ to obtain;

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= \frac{11}{21}y_n + \frac{3}{7}y_{n+1} + \frac{1}{21}y_{n+\frac{3}{2}} - \frac{5}{36}h^2 f_{n+\frac{1}{2}} - \frac{1}{252}h^2 f_{n+2} \\ y_{n+\frac{3}{2}} &= \frac{13}{37}y_n - \frac{63}{37}y_{n+\frac{1}{2}} + \frac{87}{37}y_{n+1} - \frac{35}{148}h^2 f_{n+\frac{3}{2}} - \frac{11}{148}h^2 f_{n+2} \end{aligned}\right\} \quad (12)$$

.3.1 Order and Error Constants

Following the work of Mohammed(2015),the Local Truncation Error (LTE) for a block method of the form (3) is defined with the linear operator;

$$\mathcal{L}[y(x),h] = \sum_{j=0}^{k}\Big[\alpha_j y(x+jh) - \alpha_u(x)y(x+uh) - \alpha_v(x)y(x+vh) - h^2\beta_k(x)f_{n+k}\Big] \quad (13)$$

Assuming that $y(x)$ is sufficiently differentiable such that the linear operator defined above can be expanded as a Taylor's series about the point x. Then,

$$\mathcal{L}[y(x),h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_q h^q y^q(x) + \cdots \quad (14)$$

The method above will be consistent if $\mathcal{L}[y(x),h] \to 0$ as $h \to 0$. Therefore, we can compare the coefficients to have

$$\left.\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k = \sum_{j=0}^{k} \alpha_j \\ C_1 &= (\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k) = \sum_{j=0}^{k} (j\alpha_j - \beta_j) \\ &\vdots \\ C_q &= \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k) \end{aligned}\right\} \quad (15)$$

The method is consistent if $C_0 = C_1 = \cdots = C_{p+1} = 0$, for $C_{p+2} \neq 0$. The constant C_{p+2} is the error constant. From our computation, the 2BHBDF has order $p = 3$ and the error constant is $C_5 = \left(-\frac{1}{112}, \frac{19}{1120}, -\frac{31}{6720}, \frac{1}{280}, -\frac{17}{2240}, -\frac{83}{3360}, \frac{5}{2688}, \frac{145}{14208}\right)^T$. Therefore, the 2BHBDF is consistent since each discrete scheme has order greater than one.

3.2 Zero Stability

According to Awari (2017), a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root with modulus one, it is simple, i.e. $|r| \leq 1$ and has multiplicity not greater than the order of the differential equation.

To obtain the zero-stability of 2BHBDF, we shall express the proposed methods in matrix difference equation form

$$P^{(1)}Y_{\omega+1} = P^{(0)}Y_\omega + h^2Q^{(0)}F_\omega + h^2Q^{(1)}F_{\omega+1} \quad (16)$$

Where

$$Y_{\omega+1} = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} Y_{\omega} = \begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_{n} \end{pmatrix}$$

$$F_{\omega+1} = \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} f_{\omega} = \begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_{n} \end{pmatrix}$$

$P^{(1)}$,$P^{(0)}$, $Q^{(1)}$, and $Q^{(0)}$ are $(k+1)\times(k+1)$ matrices obtained from the combined coefficients of the 2BHBDF. The roots of the first characteristics polynomial $\rho(r)$ is obtained from;

$$\rho(r) = \left|rP^{(1)} - P^{(0)}\right| \tag{17}$$

$$p^{(1)} = \begin{pmatrix} 1 & -\frac{3}{7} & -\frac{1}{21} & 0 \\ -\frac{56}{19} & 1 & \frac{88}{57} & 0 \\ \frac{63}{37} & -\frac{87}{37} & 1 & 0 \\ -\frac{8}{5} & \frac{114}{35} & -\frac{104}{35} & 1 \end{pmatrix}, \quad p^{(0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{11}{21} \\ 0 & 0 & 0 & -\frac{23}{57} \\ 0 & 0 & 0 & \frac{13}{57} \\ 0 & 0 & 0 & -\frac{11}{35} \end{pmatrix}$$

$$\rho(r) = \frac{1400}{703}r^4 - \frac{1400}{703}r^3$$

Then, $r = (0,0,0,1)$, therefore 2SBHBDF is zero-stable since $\left|r_j\right| \leq 1$.

3.3. Convergence

According to (Henrici, 1962), consistency and zero stability are sufficient conditions for a linear multistep method to be convergent, therefore, the 2BHBDF is convergent.

4.0 Numerical Experiments

In this section, some numerical experiments of linear, non-linear, homogeneous, inhomogeneous and system of second-order ordinary differential equations are solved using the derived 2BHBDF

Problem 1: Variable Coefficient Linear Type:

$$t^2\frac{d^2y(t)}{dt^2}+\frac{3}{2}t\frac{dy(t)}{dt}-\frac{1}{2}y(t)=0, \quad y(1)=2, y'(1)=5$$

Exact Solution: $y(t)=\frac{14}{3}\sqrt{t}-\frac{8}{3t}$

Table 4.1: Comparison of Exact Solution and 2BHBDF for Problem 1 for $h = 0.01$

t	Exact	2BHBDF	Error
0.0	2.0000000000000000000	2.0000000000000000000	$0.00000000000 \times 10^{0}$
0.1	2.4701988672182829769	2.4701988901235716030	$2.29052886260 \times 10^{-8}$
0.2	2.8898549811593281700	2.8898550475903798431	$6.64310516730 \times 10^{-8}$
0.3	3.2695365991805926204	3.2695367183658320765	$1.191852394561 \times 10^{-7}$
0.4	3.6169125594644035444	3.6169127348535730639	$1.753891695194 \times 10^{-7}$
0.5	3.9376982887163044513	3.9376985207858306440	$2.320695261927 \times 10^{-7}$
0.6	4.2362516323143080864	4.2362519200216870588	$2.877073789723 \times 10^{-7}$
0.7	4.5159614605420799766	4.5159618021000866133	$3.415580066366 \times 10^{-7}$
0.8	4.7795088555179296682	4.7795092488135041059	$3.932955744375 \times 10^{-7}$
0.9	5.0290473123789455970	5.0290477552008961247	$4.428219505277 \times 10^{-7}$
1.0	5.2663299577411102278	5.2663304479025818024	$4.901614715746 \times 10^{-7}$

Problem 2: Variable Coefficient Non-Linear Type: $\frac{d^2y(t)}{dt^2} = t\left(\frac{dy(t)}{dt}\right)^2;\ y(0) = 1, y'(0) = \frac{1}{2}$

Exact Solution: $y(t) = 1 + \frac{1}{2}\ln\left(\frac{2+t}{2-t}\right)$

Table 4.2: Comparison of Exact Solution and 2BHBDF for Problem 2 for $h = 0.01$

t	Exact	2BHBDF	Error
0.0	1.0000000000000000000	1.0000000000000000000	$0.00000000000 \times 10^{0}$
0.1	1.0500417292784912682	1.0500417293457835468	$6.729227860 \times 10^{-1}$
0.2	1.1003353477310755806	1.1003353479770131443	$2.459375637 \times 10^{-10}$
0.3	1.1511404359364668053	1.1511404364899374194	$5.534706141 \times 10^{-10}$
0.4	1.2027325540540821910	1.2027325550756534539	$1.0215712629 \times 10^{-9}$
0.5	1.2554128118829953416	1.2554128135847063960	$1.7017110544 \times 10^{-9}$
0.6	1.3095196042031117155	1.3095196068782244240	$2.6751127085 \times 10^{-9}$
0.7	1.3654437542713961691	1.3654437583415579519	$4.0701617828 \times 10^{-9}$

0.8	1.4236489301936018069	1.4236489362870485254	$6.0934467185 \times 10^{-9}$
0.9	1.4847002785940517416	1.4847002876810812149	$9.0870294733 \times 10^{-9}$
1.0	1.5493061443340548457	1.5493061579730394425	$1.36389845968 \times 10^{-8}$

5.0 Conclusion

In this paper, the procedure of interpolation and collocation is applied to derive a 2-step block hybrid backward differentiation formula for solving second order differential equations. In the tables **4.1 – 4.2,** we compared the new method with the exact solution of some numerical examples after establishing the basic numerical properties of consistent and convergent and found to agree with the exact solution with relatively small errors.

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