Lapai Journal of Science and Technology, vol. 2, No.2 (2014)

GENERALIZATION OF TAU APPROXIMANT AND ERROR ESTIMATE OF INTEGRAL FORM OF TAU METHODS FOR SOME CLASS OF ORDINARY DIFFERENTIAL EQUATIONS

***Ma'ali , A .I., [2]Adeniyi, R. B., [1]Badeggi, A. Y. and [3]Mohammed, U.**

[*1]Department of Mathematics and Computer Science, Ibrahim Badamasi Babangida University, Lapai, Niger State, Nigeria.

[2] Department of Mathematics, University of Ilorin, Ilorin. Nigeria

[3]Department of Mathematics, Federal University of Technology, Minna. Niger State. Nigeria.

ABSTRACT

This paper focuses on concern with the generalization of tau approximants and their corresponding error estimates for some class of ordinary differential equations (ODEs) characterized by $m + s = 3(i.e\ for\ m = 1, s = 2, m = 2, s = 1\ and\ m = 3, s = 0)$ where m and s are the order of differential equations and number of overdetermination, respectively. The results obtained were validated with some numerical examples.

Keywords: Tau method, variant, approximant, error estimate, overdetermination.

INTRODUCTION

The tau method first introduced by Lanczos in 1938 has over time been developed into different variants so as to either improve its accuracy, widen its scope of application or render it amenable for easier use.In this direction, Lanczos (1956) developed a modification based on the use of canonical polynomials and Ortiz (1969) proposed a recursive generation of these polynomials to give some flexibility in the procedures involved. Ortiz (1974) discussed the procedure in the frame work of graph theory. On the basis of these results, Ortiz (1974) showed that the elements of the canonical polynomial sequence can be generated by means of a simple recursion relation which is self-starting and explicit. With the aid of a certain procedure which Lanczos (1956) called "tau method", approximation of high accuracy could be obtained for a number of functions used in scientific and engineering computations (See Luke (1969), Ortiz (1974)). Adeniyi (1991) reported a generalized tau method based on the original formulation, the first variant of the tau method, otherwise referred to as the differential form and which also incorporated an error estimation of the tau method with extension to segmented or piece-wise tau approximation and solution to nonlinear problems.

INTEGRAL FORMULATION OF THE TAU METHOD

The method was developed by Lanczos (1938) for the solution of m-th order problem:-

$$Ly(x) \equiv \sum_{r=0}^{m} P_r(x) y^{(r)}(x) = \sum_{r=0}^{n} f_r x^r, \quad a \le x \le b \qquad (1.1a)$$

$$L^* y(x) \equiv \sum_{r=0}^{m} a_{rk}(x) y^{(r)}(x_{rk}) = \rho_k \qquad (1.1b)$$

where $y^{(r)}(x)$ is the derivatives of order $y(x)$ and $f(x)$, and $P_r(x)$, r = 0(1)m , are polynomials of given functions, where a_{rk} , x_{rk} and ρ_k are given real numbers .

The method seeks an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r \qquad (1.2)$$

which satisfies the perturbed problem

$$Ly_n(x) \equiv \sum_{r=0}^{m} P_r(x) y_n(x)^{(r)}(x) = \sum_{r=0}^{n} f_r x^r + H_n(x) \qquad (1.3)$$

where,

$$H_n(x) \equiv \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+2}(x) \qquad (1.4)$$

is the chebyshev polynomials in the interval [a,b] and τ_s' are fixed tau parameters to be determined together with the coefficients of $y_n(x)$ where

$$S = \max\{N_r - r : 0 \le r \le m\} \ge 0 \qquad (1.5)$$

is the number of overdetermination of (1.1) and

$$T_n(x) = cos\{x\, cos^{-1}\left(\frac{2x-a-b}{a-b}\right)\} \equiv \sum_{r=1}^{n} C_r^{(n)} x^r \qquad (1.6)$$

is the n-th shifted chebyshev polynomial in the interval [a,b].

The integrated form of the tau method arose as an attempt to improve the accuracy of the approximant $y_n(x)$ of $y(x)$, resulting from differential formulation whereby we integrate (1.1) to obtain

$$I_L\big(y(x)\big) \equiv \iiint \underset{....}{\overset{m}{}} \int \sum_{r=0}^{m} P_r(x) y_n(x)^{(r)}(x) dxdx \dots . dx \tag{1.7}$$

$$\equiv \iiint \underset{....}{\overset{m}{}} \int \left(\sum_{r=0}^{n} f_r x^r + \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+2}(x) \right) dxdx \dots . dx + H_{n+m}(x) \tag{1.8}$$

The tau problem (1.8) often yields or gives more accurate approximation than the differential form due to higher perturbation term $H_{n+m}(x)$

Error Estimation for the Integral Form

The integrated error estimation is aim at improving the accuracy of the estimate of the differential of the tau method. To this end, let $\iiint \underset{....}{\overset{i}{}} \int g(x) dxdx \dots dx$ denotes the indefinite integration I times applied to the function g(x) and let

$$I_L = \iiint \underset{....}{\overset{m}{}} \int L(.) dxdx \dots dx \tag{1.9}$$

The integral form of the error equation

$$Le_n(x) = -H_n(x) \tag{1.10}$$

is therefore

$$I_L\,(e_n(x))_{n+1} = -\iiint \underset{....}{\overset{m}{}} \int H_n(x) dxdx \dots dx \tag{1.11}$$

which gives the perturbed error equation of (1.11) as

$$I_L\,(e_n(x))_{n+1} = -\iiint \underset{....}{\overset{m}{}} \int \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+1}(x)\, dxdx \dots dx + C_{m-1}(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+3}(x) \tag{1.12}$$

where,

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+1}(x) \tag{1.13a}$$

and $H_{n+m}(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+3}(x)$ (1.13b)

and $C_{m-1}(x)$ arises from constant of indefinite integration which is satisfied by $(e_n(x))_{n+1}$ given by

$$(e_n(x))_{n+1} = \frac{V_m(x)\emptyset_n(x)T_{n-m+1}(x)}{C_{(n-m+1)}^{(n-m+1)}} \qquad (1.14)$$

with $\emptyset_n$ replaced by $\widehat{\emptyset}_n$, we now insert (1.14) into (1.12) and equate the coefficients of powers $x^{n+m+s+1}, x^{n+m+s}, \dots, x^{n-m}$ for the determination of $\widehat{\emptyset}_n$ of $(e_n(x))_{n+1}$.We then obtain

$$\varepsilon^* = \frac{|\widehat{\emptyset}_n|}{2^{2n-2m+1}}, \qquad (1.15)$$

an estimate of ε .

Derivation of Tau Approximant by Integral Form

In this section, we shall consider the derivation of tau approximants of integral formulation of the Lanczos Tau method for the class of problem (1.1) characterized by m+s = 3.

From (1.1) and (1.5) we have the following cases.

The Case M = 1, S = 2

From (1.1), the general case for m = 1 and s = 2 is given by:

$$Ly(x) := (P_{10} + P_{11}x + P_{12}x^2 + P_{13}x^3)y(x)' + (P_{00} + P_{01}x + P_{02}x^2)y(x) = \sum_{r=0}^{n} f_r x^r + \tau_1 T_{n+3}(x) + \tau_2 T_{n+2}(x) + \tau_3 T_{n+1}(x) . \qquad (2.1a)$$

$$y(0) = \rho_0 \qquad (2.1b)$$

Thus, we have,

$$\int_0^x (P_{10} + P_{11}t + P_{12}t^2 + P_{13}t^3)\, y(t)'\, dt + \int_0^x (P_{00} + P_{01}t + P_{02}t^2) y(t) dt = \int_0^x \left(\sum_{r=0}^{n} f_r t^r\right) dt + T_{n+3}(x) + \tau_2 T_{n+2}(x) + \tau_3 T_{n+1}(x) \qquad (2.2a)$$

Where,

$$T_{n+3}(x) = \sum_{r=0}^{n+3} C_r^{(n+3)} x^r, \quad T_{n+2}(x) = \sum_{r=0}^{n+2} C_r^{(n+2)} x^r, T_{n+1}(x) = \sum_{r=0}^{n+1} C_r^{(n+1)} x^r. \qquad (2.2b)$$

Seeking an approximate solution of the form

$$y_n(x) = \sum_{r=0}^{n} a_r x^r \qquad (2.3)$$

and substituting (2.3) into (2.2) we obtain

$$\int_0^x (P_{10} + P_{11}t + P_{12}t^2 + P_{13}t^3) y_n(t)'\, dt + \int_0^x (P_{00} + P_{01}t + P_{02}t^2) y_n(t) dt$$

$$=\int_0^x \left(\sum_{\mathrm{r}=0}^{n} f_r t^{\mathrm{r}}\right)dt + \tau_1 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r + \tau_2 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r + \tau_3 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r \qquad (2.4)$$

Integrating the terms of (2.4) and collecting the like terms we have:

$$P_{10} \sum_{r=0}^{n} a_r x^r + \sum_{r=0}^{n} \left[\frac{P_{00}+rP_{11}}{r+1}\right] a_r x^{r+1} + \sum_{r=0}^{n} \left[\frac{P_{01}+rP_{12}}{r+2}\right] a_r \mathrm{x}^{r+2} + \sum_{r=0}^{n} \left[\frac{P_{02}+rP_{13}}{r+3}\right] a_r x^{r+3}$$

$$-\tau_1 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r - \tau_2 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r - \tau_3 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r = P_{10}\rho_0 + \sum_{r=0}^{n} f_r \frac{x^{r+1}}{r+1} \qquad (2.5)$$

Equating the corresponding powers of x for n = 5 in (2.5) above, we obtain the following tau system.

$$A=\begin{pmatrix}
P_{10} & 0 & 0 & 0 & 0 & 0 & -C_0^{(8)} & -C_0^{(7)} & -C_0^{(6)} \\
P_{00} & P_{10} & 0 & 0 & 0 & 0 & -C_1^{(8)} & -C_1^{(7)} & -C_1^{(6)} \\
\frac{P_{01}}{2} & \frac{P_{00}+P_{11}}{2} & P_{10} & 0 & 0 & 0 & -C_2^{(8)} & -C_2^{(7)} & -C_2^{(6)} \\
\frac{P_{02}}{3} & \frac{P_{01}+P_{12}}{3} & \frac{P_{00}+2P_{11}}{3} & P_{10} & 0 & 0 & -C_3^{(8)} & -C_3^{(7)} & -C_3^{(6)} \\
0 & \frac{P_{02}+P_{13}}{4} & \frac{P_{01}+2P_{12}}{4} & \frac{P_{00}+3P_{11}}{4} & P_{10} & 0 & -C_4^{(8)} & -C_4^{(7)} & -C_4^{(6)} \\
0 & 0 & \frac{P_{02}+2P_{13}}{5} & \frac{P_{01}+3P_{12}}{5} & \frac{P_{00}+4P_{11}}{5} & P_{10} & -C_5^{(8)} & -C_5^{(7)} & -C_5^{(6)} \\
0 & 0 & 0 & \frac{P_{02}+3P_{13}}{6} & \frac{P_{01}+4P_{12}}{6} & \frac{P_{00}+5P_{11}}{6} & -C_6^{(8)} & -C_6^{(7)} & -C_6^{(6)} \\
0 & 0 & 0 & 0 & \frac{P_{02}+4P_{13}}{7} & \frac{P_{01}+5P_{12}}{7} & -C_7^{(8)} & -C_7^{(7)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{P_{02}+5P_{13}}{8} & -C_8^{(8)} & 0 & 0
\end{pmatrix}$$

$$\tau = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} P_{10}\rho_0 \\ f_0 \\ \frac{f_1}{2} \\ \frac{f_2}{3} \\ \frac{f_3}{4} \\ \frac{f_4}{5} \\ \frac{f_5}{6} \\ 0 \\ 0 \end{pmatrix} \qquad (2.6)$$

The case m = 2 , s = 1

From (1.1), the general case for m = 2 , s = 1 is

$$\int_0^x \int_0^u (P_{20} + P_{21}t + P_{22}t^2 + P_{23}t^3)y_n''(t)dtdu + \int_0^x \int_0^u (P_{10} + P_{11}t + P_{12}t^2)y_n'(t)dtdu +$$

$$\int_0^x \int_0^u (P_{00} + P_{01}t)y_n(t)dtdu = \int_0^x \int_0^u (\textstyle\sum_{r=0}^n f_r t^r)\, dtdu + H_{n+m}(x) \qquad (2.7a)$$

$$y(0) = \rho_0 \ , \quad y'(0) = \rho_1 \qquad (2.7b)$$

$$H_n + m(x) = \tau_1 T_{n+3}(x) + \tau_2 T_{n+2}(x) + \tau_3 T_{n+1}(x) \qquad (2.8)$$

Inserting (2.3) into (2.7), we have

$$\int_0^x \int_0^u (P_{20} + P_{21}t + P_{22}t^2 + P_{23}t^3)y_n''(t)dtdu + \int_0^x \int_0^u (P_{10} + P_{11}t + P_{12}t^2)y_n'(t)dtdu +$$

$$\int_0^x \int_0^u (P_{00} + P_{01}t + P_{02}t^2)y_n(t)dtdu = \int_0^x \int_0^u (\textstyle\sum_{r=0}^n f_r t^r)\, dtdu + \tau_1 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r +$$

$$\tau_2 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r + \tau_3 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r \qquad (2.9)$$

Integrating the terms of (2.9) yields,

$$P_{20} \sum_{r=0}^n a_r x^r + \sum_{r=0}^n \left[\frac{P_{10}+(r-1)P_{21}}{r+1}\right] a_r x^{r+1} + \sum_{r=0}^n \left[\frac{P_{00}+rP_{11}+r(r-1)P_{22}}{(r+1)(r+2)}\right] a_r x^{r+2} +$$

$\sum_{r=0}^{n}\left[\frac{P_{01}+rP_{12}+r(r-1)P_{23}}{(r+2)(r+3)}\right]a_r x^{r+3}$ - $\tau_1\sum_{r=0}^{n+3}C_r^{(n+3)}x^r-\tau_2\sum_{r=0}^{n+2}C_r^{(n+2)}x^r$ - $\tau_3\sum_{r=0}^{n+1}C_r^{(n+1)}x^r$

$$= P_{20}\rho_0 + P_{20}\rho_1 + (P_{10}-P_{21})\rho_0 + \sum_{r=0}^{n} f_r \frac{x^{r+2}}{(r+1)(r+2)} \qquad (2.10)$$

This gives the following tau system for n = 5 by equating the corresponding coefficients of powers of x in (2.10)

A =

$$\begin{pmatrix}
P_{20} & 0 & 0 & 0 & 0 & 0 & -C_0^{(8)} & -C_0^{(7)} & -C_0^{(6)} \\
P_{10}-P_{21} & P_{20} & 0 & 0 & 0 & 0 & -C_1^{(8)} & -C_1^{(7)} & -C_1^{(6)} \\
\frac{P_{00}}{2} & \frac{P_{10}}{2} & P_{20} & 0 & 0 & 0 & -C_2^{(8)} & -C_2^{(7)} & -C_2^{(6)} \\
\frac{P_{01}}{6} & \frac{P_{00}+P_{11}}{6} & \frac{P_{10}+P_{21}}{3} & P_{20} & 0 & 0 & -C_3^{(8)} & -C_3^{(7)} & -C_3^{(6)} \\
0 & \frac{P_{01}+P_{12}}{12} & \frac{P_{00}+2P_{11}+2P_{22}}{12} & \frac{P_{10}+2P_{21}}{4} & P_{20} & 0 & -C_4^{(8)} & -C_4^{(7)} & -C_4^{(6)} \\
0 & 0 & \frac{P_{01}+2P_{12}+2P_{23}}{20} & \frac{P_{00}+3P_{11}+6P_{22}}{20} & \frac{P_{10}+3P_{21}}{5} & P_{20} & -C_5^{(8)} & -C_5^{(7)} & -C_5^{(6)} \\
0 & 0 & 0 & \frac{P_{01}+3P_{12}+6P_{23}}{30} & \frac{P_{00}+4P_{11}+12P_{22}}{30} & \frac{P_{10}+4P_{21}}{6} & -C_6^{(8)} & -C_6^{(7)} & -C_6^{(6)} \\
0 & 0 & 0 & 0 & \frac{P_{01}+4P_{12}+12P_{23}}{42} & \frac{P_{00}+5P_{11}+20P_{22}}{42} & -C_7^{(8)} & -C_7^{(7)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{P_{01}+5P_{12}+20P_{23}}{56} & -C_8^{(8)} & 0 & 0
\end{pmatrix}$$

$$\tau = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} P_{20}\rho_0 \\ P_{20}\rho_1 + (P_{10}-P_{21})\rho_0 \\ \frac{f_0}{2} \\ \frac{f_1}{6} \\ \frac{f_2}{12} \\ \frac{f_3}{20} \\ \frac{f_4}{30} \\ \frac{f_5}{42} \\ 0 \end{pmatrix} \qquad (2.11)$$

The case m = 3, s = 0

From (1.1), for m = 3 , s = 0 we have:

$$(P_{30}+P_{31}x+P_{32}x^2+P_{33}x^3)y'''(x)+(P_{20}+P_{21}x+P_{22}x^2)y''(x)+$$
$$(P_{10}+P_{11}x)y'(x)+P_{00}y(x)=\sum_{r=0}^{n} f_r x^r+H_{n+m}(x) \qquad (2.12a)$$

$$y(0)=\rho_0\ ,\ y'(0)=\rho_1\ ,\ y''(0)=\rho_2 \qquad (2.12b)$$

which yields ,

$$\int_0^x\int_0^u\int_0^t(P_{30}+P_{31}w+P_{32}w^2+P_{33}w^3)y_n'''(w)dwdtdu+\int_0^x\int_0^u\int_0^t(P_{20}+P_{21}w+$$

$$P_{22}w^2)\ y_n''(w)dwdtdu+\int_0^x\int_0^u\int_0^t(P_{10}+P_{11}w)\ y_n'(w)_{n+1}dwdtdu\ +$$

$$\int_0^x\int_0^u\int_0^t P_{00}\,y_n(w)dwdtdu=\int_0^x\int_0^u\int_0^t\sum_{r=0}^{n} f_r x^r+H_{n+m}(x) \qquad (2.13)$$

where $H_{n+m+1}(x)$ is as given in (2.8).Using the same procedure, we obtain the following tau system:

$$A=\begin{bmatrix} P_{30} & 0 & 0 & 0 & 0 & 0 & -C_0^{(8)} & -C_0^{(7)} & -C_0^{(6)} \\ Z_{21} & P_{30} & 0 & 0 & 0 & 0 & -C_1^{(8)} & -C_1^{(7)} & -C_1^{(6)} \\ Z_{31} & Z_{32} & P_{30} & 0 & 0 & 0 & -C_2^{(8)} & -C_2^{(7)} & -C_2^{(6)} \\ Z_{41} & Z_{42} & Z_{43} & P_{30} & 0 & 0 & -C_3^{(8)} & -C_3^{(7)} & -C_3^{(6)} \\ 0 & Z_{52} & Z_{53} & Z_{54} & P_{30} & 0 & -C_4^{(8)} & -C_4^{(7)} & -C_4^{(6)} \\ 0 & 0 & Z_{63} & Z_{64} & Z_{65} & P_{30} & -C_5^{(8)} & -C_5^{(7)} & -C_5^{(6)} \\ 0 & 0 & 0 & Z_{74} & Z_{75} & Z_{76} & -C_6^{(8)} & -C_6^{(7)} & -C_6^{(6)} \\ 0 & 0 & 0 & 0 & Z_{85} & Z_{86} & -C_7^{(8)} & -C_7^{(7)} & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_{96} & -C_8^{(8)} & 0 & 0 \end{bmatrix}$$

$$\tau = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} P_{30}\rho_0 \\ P_{30}\rho_1 + (P_{20} - 2P_{31})\rho_0 \\ \frac{P_{30}}{2}\rho_2 + \left(\frac{P_{20}}{2} - \frac{P_{31}}{2}\right)\rho_1 + \left(\frac{P_{10}}{2} - \frac{P_{21}}{2} + P_{32}\right)\rho_0 \\ \frac{f_0}{6} \\ \frac{f_1}{24} \\ \frac{f_2}{60} \\ \frac{f_3}{120} \\ \frac{f_4}{210} \\ \frac{f_5}{336} \end{pmatrix} \quad (2.14)$$

where,

$Z_{21} = P_{20} - 2P_{31}$, $Z_{31} = \frac{P_{10} - P_{21} + P_{32}}{2}$, $Z_{41} = \frac{P_{00}}{6}$, $Z_{32} = \frac{P_{20}+P_{31}}{2}$, $Z_{42} = \frac{P_{10}}{6}$

$Z_{52} = \frac{P_{00}+P_{11}}{24}$, $Z_{43} = \frac{P_{20}}{3}$, $Z_{53} = \frac{P_{10}+P_{21}}{12}$, $Z_{63} = \frac{P_{00}+2P_{11}+2P_{22}}{60}$, $Z_{54} = \frac{P_{20}+P_{31}}{4}$

$Z_{64} = \frac{P_{10}+2P_{21}+2P_{32}}{20}$, $Z_{74} = \frac{P_{00}+3P_{11}+6P_{22}+6P_{33}}{120}$, $Z_{65} = \frac{P_{20}+2P_{31}}{5}$, $Z_{75} = \frac{P_{10}+3P_{21}+6P_{32}}{30}$

$Z_{85} = \frac{P_{00}+4P_{11}+12P_{22}+24P_{33}}{210}$, $Z_{76} = \frac{P_{20}+3P_{31}}{6}$, $Z_{86} = \frac{P_{10}+4P_{21}+12P_{32}}{42}$

$Z_{96} = \frac{P_{00}+5P_{11}+20P_{22}+60P_{33}}{336}$

Thus, we obtained the following expressions for $a'_{i,j}s$ and $b'_i s$ i.e

$a_{kk} = P_{m0}$, $\forall\ k = 1(1)(n+1)$, $\forall m$

$$a_{kr} = \begin{cases} \dfrac{\sum_{k=0}^{1} k!\binom{r-m}{k} P_{m-2+k,k}}{r_{p_1}}, & \forall\, k = 2(1)(n+2), \forall r = 1(1)(n+1), \quad \forall\, m = 1,2,3\,. \\ \dfrac{\sum_{k=0}^{2} k!\binom{r-m+1}{k} P_{m-2+k,k}}{(r+1)_{p_2}}, & \forall\, k = 3(1)(n+3), \forall r = 1(1)(n+1), \forall\, m = 2,3. \\ \dfrac{\sum_{k=0}^{3} k!\binom{r-m+2}{k} P_{m-3+k,k}}{(r+2)_{p_3}}, & \forall\, k = 4(1)(n+4), \forall r = 1(1)(n+1), \quad \forall\, m = 3. \\ \vdots & \\ \dfrac{\sum_{k=0}^{m} k!\binom{r-1}{k} P_{kk}}{(r+m-1)_{P_m}}, & \forall\, k = (m+1)(1)(n+m+1), \quad r = 1(1)(n+1), \quad \forall\, m \\ \dfrac{\sum_{k=0}^{m} k!\binom{r-1}{k} P_{k,k+1}}{(r+m)_{P_m}}, & \forall\, k = (m+s+1)(1)(n+m+2), \quad r = 1(1)(n+1), \forall\, m \\ \dfrac{\sum_{k=0}^{m} k!\binom{r-1}{k} P_{k,k+2}}{(r+m+1)_{P_m}}, & \forall\, k = (m+s+1)(1)(n+m+s), r = 1(1)(n+1), \quad \forall\, m \\ \vdots & \\ \dfrac{\sum_{k=0}^{m} k!\binom{r-1}{k} P_{k,k+s}}{(r+m+s-1)_{P_m}}, & \forall\, k = (m+s+1)(1)(n+m+s), \forall r = 1(1)(n+1), \quad \forall\, m \end{cases}$$

and

$$a_{kr} = \begin{cases} 0, & \forall\, r > k,\; k = 1(1)(n),\; r = 2(1)(n+1) \\ \\ 0, & \forall\; (m+s+2)(1)(n+m+s+1)\;, r = 1(1)n \end{cases}$$

$$a_{k,n+2} = -C_{k-1}^{(n+m+s)}\;,\quad k = 1(1)(n+m+s+1)$$

$$a_{k,n+3} = -C_{k-1}^{(n+m+s-1)}\;,\quad k = 1(1)(n+m+s)$$

$a_{k,n+4} =$ -$C_{k-1}^{(n+m+s-2)}$, $k = 1(1)(n+m+s-1)$

$a_{k,n+m+s+1} =$ - $C_{k-1}^{(n+s-1)}$, $k = 1(1)(n+s)$ and $b_1 = P_{\mathrm{m}0}$, ∀ m

$b_2 = \frac{1}{(m-1)!}\{\alpha_1 \sum_{r=0}^{0}(-1)^r r!\,P_{r+2,r} + \alpha_0 \sum_{r=0}^{1}(-1)^r r!\,P_{r+1,r}\}$ ∀ m = 2

$b_3 = \frac{1}{(m-1)!}\{\alpha_2 \sum_{r=0}^{0}(-1)^r r!\,P_{r+3,r} + \alpha_1 \sum_{r=0}^{1}(-1)^r r!\,P_{r+2,r} +$
$\alpha_0 \sum_{r=0}^{1}(-1)^r r!\,P_{r+1,r}\}$ ∀ m

$3b_{\mathrm{i}} = \frac{f_{i-\mathrm{m}-1}}{\prod_{\mathrm{r}=1}^{\mathrm{m}}(\mathrm{i}-\mathrm{r})}$, ∀ $\mathrm{m}+1 \le i \le 2n-\mathrm{s}$, $1 \le m \le 3$

Derivation of Error Estimation for Integral Form

In this section, we shall derive the error estimates for the cases characterized by m+s = 3 and deduce the general error estimate that captures all the cases.

The case m = 1, s = 2

From (1.1), we have.

$$I_L\big(e_n(x)\big)_{n+1} := \int_0^x (P_{10}+P_{11}t+P_{12}t^2+P_{13}t^3)\,\big(e_n'(x)\big)_{n+1}dt + \int_0^x (P_{00}+P_{01}t+ P_{02}t^2)\,\big(e_n(x)\big)_{n+1}dt = -\tau_1 \int_0^x \sum_{r=0}^{n+2} C_r^{(n+2)} t^r\,dt - \tau_2 \int_0^x \sum_{r=0}^{n+1} C_r^{(n+1)} t^r\,dt + -\tau_3 \int_0^x \sum_{r=0}^{n+1} C_r^{(n+1)} t^r\,dt + \widetilde{\tau_1} \int_0^x \sum_{r=0}^{n+4} C_r^{(n+4)} x^r + \widetilde{\tau_2} \int_0^x \sum_{r=0}^{n+3} C_r^{(n+3)} x^r \; \widetilde{\tau_1} \int_0^x \sum_{r=0}^{n+2} C_r^{(n+2)} x^r \qquad (3.1)$$

where,

$$(e_n(x))_{n+1} = \frac{\phi_n T_n(x)}{C_n^{(n)}} = \frac{\phi_n \sum_{\mathrm{r}=0}^{\mathrm{n}} \mathrm{C}_{\mathrm{r}}^{(\mathrm{n})} \mathrm{x}^{\mathrm{r}}}{C_n^{(n)}}$$

i.e, $(e_n(x))_{n+1} = \frac{\phi_n}{k_1}\{k_1 x^{n+1} + k_2 x^n + k_3 x^{n-1} + + + \cdots\}$ (3.2)

where,

$k_1 = C_n^{(n)}$, $k_2 = C_{n-1}^{(n)}$, $k_2 = C_{n-2}^{(n)}$ e.tc. and

$$\int_0^x (e_n(t))_{n+1}\,dt = \frac{\phi_n}{k_1}\left\{\frac{k_1 x^{n+2}}{n+2} + \frac{k_2 x^{n+1}}{n+1} + \frac{k_3 x^n}{n} + + + \cdots\right\} \qquad (3.3)$$

Substituting (3.2) and (3.3) into (3.1) and equating the corresponding coefficients of

x^{n+4}, x^{n+3}, x^{n+2} and x^{n+1} we have,

$$\hat{\tau}_1 C_{n+4}^{(n+4)} = \frac{\phi_{n_{\lambda_1}}}{k_1} \quad (3.4a)$$

$$\hat{\tau}_1 C_{n+3}^{(n+4)} + \hat{\tau}_2 C_{n+3}^{(n+3)} - \frac{\tau_1 C_{n+2}^{(n+2)}}{n+3} = \frac{\phi_{n_{\lambda_2}}}{k_1} \quad (3.4b)$$

$$\hat{\tau}_1 C_{n+2}^{(n+4)} + \hat{\tau}_2 C_{n+2}^{(n+3)} + \hat{\tau}_3 C_{n+2}^{(n+2)} - \frac{\tau_1 C_{n+1}^{(n+2)}}{n+2} - \frac{\tau_2 C_{n+1}^{(n+1)}}{n+2} = \frac{\phi_{\lambda_3}}{k_1} \quad (3.4c)$$

$$\hat{\tau}_1 C_{n+1}^{(n+4)} + \hat{\tau}_2 C_{n+1}^{(n+3)} + \hat{\tau}_3 C_{n+1}^{(n+2)} - \frac{\tau_1 C_{n}^{(n+2)}}{n+1} - \frac{\tau_2 C_{n}^{(n+1)}}{n+1} - \frac{\tau_3 C_{n}^{(n)}}{n+1} = \frac{\phi_{n_{\lambda_4}}}{k_1} \quad (3.4d)$$

Solving the equations (3.4a)- (3.4d) by forward substitution and using well – known relations $C_n^{(n)} = 2^{2n-1}, C_{n-1}^{(n)} = \frac{-1}{2} n C_n^{(n)}, C_{n-1}^{(n)} = -n2^{2n-2}$, we have the following value for ϕ_n.

$$\phi_n = \left\{ \frac{C_{n+1}^{(n+3)} C_{n+2}^{(n+2)}}{(n+3) C_{n+3}^{(n+3)}} - \frac{C_n^{(n+2)}}{(n+1)} \right\} \frac{K_1 \tau_1}{R_4} - \frac{K_1^2 \tau_3}{(n+1) R_4} \quad (3.5)$$

With the following recursive form

$$R_1 = \lambda_1, \quad R_2 = \lambda_2 - \frac{C_{n+3}^{(n+4)} R_1}{C_{n+4}^{(n+4)}}, \quad R_3 = \lambda_3 - \frac{C_{n+2}^{(n+4)} R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+2}^{(n+3)} R_2}{C_{n+3}^{(n+3)}},$$

$$R_4 = \lambda_4 - \frac{C_{n+1}^{(n+4)} R_1}{C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+3)} R_2}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_3}{C_{n+2}^{(n+2)}} \quad (3.6a)$$

and.

$$\lambda_1 = \left[\frac{P_{02} + (n+1) P_{13}}{(n+4)} \right] K_1, \quad \lambda_2 = \left[\frac{P_{01} + (n+1) P_{12}}{(n+3)} \right] K_1 + \left[\frac{P_{02} + n P_{13}}{(n+3)} \right] K_2,$$

$$\lambda_3 = \left[\frac{P_{00} + (n+1) P_{11}}{(n+2)} \right] K_1 + \left[\frac{P_{01} + n P_{12}}{(n+2)} \right] K_2 + \left[\frac{P_{02} + (n-1) P_{13}}{(n+2)} \right] K_3$$

$$\lambda_4 = \mathrm{P}_{10} \mathrm{K}_1 + \left[\frac{P_{00} + \mathrm{n} P_{11}}{(n+1)} \right] K_2 \left[\frac{P_{01} + (n-1) P_{12}}{(n+1)} \right] K_3 + \left[\frac{P_{02} + (n-2) P_{13}}{(n+2)} \right] K_4 \quad (3.6b)$$

For m = 2, s = 1

In this case we have from (1.1)

$$\int_0^x\int_0^u(P_{20}+P_{21}t+P_{22}t^2+P_{23}t^3)\left(e_n''(t)\right)_{n+1}dtdu+\int_0^x\int_0^u(\int_0^x\int_0^u(P_{10}+P_{11}t+P_{12}t^2)(e_n'(t))_{n+1}dtdu+\int_0^x\int_0^u(P_{00}+P_{01}t)(e_n(t))_{n+1}dtdu=-\int_0^x\int_0^u(H_n(t)dtdu+H_{m+n+1}(x) \quad (3.7)$$

$$(e_n(x))_{n+1}=\frac{\phi_n(x-\propto)^m T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}}=\frac{\phi_n x^2 T_{n-1}(x)}{C_{n-1}^{(n-1)}}=\frac{\phi_n x^2 T_{n-1}(x)}{2^{2n-3}} \quad (3.8)$$

that is,

$$(e_n(x))_{n+1}=\frac{\phi_n}{k_1}\{k_1x^{n+1}+k_2x^n+k_3x^{n-1}+++\cdots\} \quad (3.9a)$$

and,

$$\int_0^x\int_0^u((e_n(t))_{n+1}dtdu=\frac{\hat{\phi}_n}{k_1}\left\{\frac{k_1x^{n+3}}{(n+2)(n+3)}+\frac{k_2x^{n+2}}{(n+1)(n+2)}+\frac{k_3x^{n+1}}{n(n+1)}+++\cdots\right\} \quad (3.9b)$$

$$\int_0^x\int_0^u(\left(e_n''(t)\right)_{n+1}dtdu=(e_n(x))_{n+1}=\frac{\hat{\phi}_n}{K_1}\{k_1x^{n+1}+k_2x^n+k_3x^{n-1}++\cdots\} \quad (3.9c)$$

$$\int_0^x\int_0^u((e_n'(t))_{n+1}dtdu=\frac{\hat{\phi}_n}{k_1}\left\{\frac{k_1x^{n+2}}{(n+2)}+\frac{k_2x^{n+1}}{(n+1)}+\frac{k_3x^n}{n}+++\cdots\right\} \quad (3.9d)$$

Continuing with the process by substituting (3.8), (3.9a) – (3.9d), collecting the like terms and equating the corresponding coefficients of $x^{n+4}, x^{n+3}, x^{n+2}$ *and* x^{n+1},we obtain the following expression for ϕ_n

$$\phi_n=\left\{\frac{C_{n+1}^{(n+3)}C_{n+1}^{(n+1)}}{(n+2)(n+3)C_{n+3}^{(n+3)}}-\frac{C_{n-1}^{(n+1)}}{n(n+1)}\right\}\frac{K_1\tau_1}{R_4}-\frac{K_1^2\tau_3}{n(n+1)R_4} \quad (3.10)$$

Where R is obtained recursively as in (3.6a) and

$$\lambda_1=\left[\frac{P_{01}+(n+1)P_{12}+\mathrm{n}(\mathrm{n}+1)P_{23}}{(\mathrm{n}+3)(n+4)}\right]K_1,\ \lambda_2=\left[\frac{P_{00}+(n+1)P_{11}+\mathrm{n}(n+1)P_{22}}{(\mathrm{n}+2)(n+3)}\right]K_1+\left[\frac{P_{01}+\mathrm{n}P_{12}+\mathrm{n}(\mathrm{n}-1)P_{23}}{(\mathrm{n}+2)(n+3)}\right]K_2\ ,$$

$\lambda_3 = \left[\frac{P_{10}+nP_{21}}{(n+2)}\right] K_1 + \left[\frac{P_{00}+\text{n}P_{11}+\text{n}(n-1)P_{22}}{(\text{n}+1)(n+2)}\right] K_2 + \left[\frac{P_{01}+(\text{n}-1)P_{12}+(\text{n}-1)(\text{n}-2)P_{23}}{(\text{n}+1)(n+2)}\right] K_3$

$$\lambda_4 = \text{P}_{20}\text{K}_1 + \left[\frac{P_{10}+(n-1)P_{21}}{(n+1)}\right] K_2 + \left[\frac{P_{00}+(\text{n}-1)P_{11}+(\text{n}-1)(n-2)P_{22}}{\text{n}(\text{n}+1)(n+2)}\right] K_3 + \left[\frac{P_{01}+(\text{n}-2)P_{12}+(\text{n}-2)(\text{n}-3)P_{23}}{\text{n}(\text{n}+1)}\right] K_4 \qquad (3.11)$$

$k_1 = C_{n-1}^{(n-1)}$, $k_2 = C_{n-2}^{(n-1)}$, $k_3 = C_{n-3}^{(n-1)}$, $k_4 = C_{n-4}^{(n-1)}$ e.t.c

For m = 3, s = 0

The general form for m =3, s = 0 from (1.1) is

$$\int_0^x \int_0^u \int_0^t (P_{30} + P_{31}w + P_{32}w^2 + P_{33}w^3)(e_n'''(w))_{n+1} dwdtdu + \int_0^x \int_0^u \int_0^t (P_{20} + P_{21}w + P_{22}w^2)\,(e_n''(w))_{n+1} dwdtdu + \int_0^x \int_0^u \int_0^t (P_{10} + P_{11}w)\,(e_n'(w))_{n+1} dwdtdu + \int_0^x \int_0^u \int_0^t P_{00}\,(e_n(w))_{n+1} dwdtdu = -\int_0^x \int_0^u \int_0^t H_n(w) dwdtdu + \widetilde{H}_{n+m+1}(\text{x}) \qquad (3.12)$$

where,

$$(e_n(x))_{n+1} = \frac{\phi_n(x-\propto)^m T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{\phi_n x^3 T_{n-2}(x)}{C_{n-2}^{(n-2)}} = \frac{\phi_n x^3 T_{n-2}(x)}{2^{2n-5}} \qquad (3.13a)$$

$$H_n(x) = \tau_1 T_{n+1}(x) + \tau_2 T_n(x) + \tau_3 T_{n-1}(x) + \tau_4 T_{n-2}(x)$$

$$\widetilde{H}_{n+m+1} = \tilde{\tau}_1 T_{n+5}^*(x) + \tilde{\tau}_2 T_{n+4}^*(x) + \tilde{\tau}_3 T_{n+3}^*(x) + \tilde{\tau}_4 T_{n+2}^*(x) \qquad (3.13b)$$

After substituting (3.13a) and (3.13b) into (3.12) and solving the resulting equations we obtained the value

of ϕ_n *as*

$$\phi_n = \left\{\frac{C_{n+1}^{(n+3)} C_n^{(n)}}{(n+1)(n+2)(n+3)C_{n+3}^{(n+3)}} - \frac{C_{n-2}^{(n)}}{(n-1)n(n+1)}\right\}\frac{K_1\tau_1}{R_4} - \frac{K_1^2\tau_3}{(n-1)n(n+1)R_4} \qquad (3.14)$$

where R is obtained recursively as in (3.6a) and

$$\lambda_1 = \left[\frac{P_{00}+(n+1)P_{11}\ +\text{n}(n+1)P_{22}+(n+1)n(n-1)P_{33}}{(n+2)(n+3)(n+4)}\right]K_1$$

$$\lambda_2 = \left[\frac{P_{10}+\text{n}P_{21}+\text{n}(n-1)P_{32}}{(n+2)(n+3)}\right]K_1\ \left[\frac{P_{00}+\text{n}P_{11}\ +\text{n}(n-1)nP_{22}+\text{n}(n-1)(n-2)P_{33}}{(n+1)(n+2)(n+3)}\right]K_1$$

$$\lambda_3 = \left[\frac{P_{20}+(n-1)P_{31}}{(n+2)}\right]K_1\ \left[\frac{P_{10}+(n-1)P_{21}+(n-1)(n-2)P_{32}}{(n+1)(n+2)}\right]K_2 + \left[\frac{P_{00}+(n-1)P_{11}+(n-1)(n-2)P_{22}+(n-1)(n-2)(n-3)P_{33}+}{n(n+1)(n+2)}\right]K_3$$

$$\lambda_4 = \text{P}_{30}\text{K}_1 + \left[\frac{P_{20}+(n-1)P_{31}}{(n+1)}\right]K_2 + \left[\frac{P_{10}+(n-2)P_{21}+(n-2)(n-3)P_{32}}{\text{n}(n+1)}\right]K_3 +$$

$$\left[\frac{P_{00}+(n-2)P_{11}+(n-2)(n-3)P_{22}+(n-2)(n-3)(n-4)P_{33}+}{n(n-1)(n-2)}\right]K_4 +$$

$$k_1 = C_{n-2}^{(n-2)}\ ,\ \ k_2 = C_{n-3}^{(n-2)}\ ,k_3 = C_{n-4}^{(n-2)}\ \ ,k_4 = C_{n-5}^{(n-2)}\quad \text{e.t.c} \qquad (3.15)$$

Thus, we observed that the expression for the ϕ_n was the same for the groupings. ie for m = 1, s = 2, m = 2, s = 1, m = 3, s = 0 (m + s = 3). Consequently, from (3.5), (3.10) and (3.14) the general expression for ϕ_n is

$$\phi_n = \left\{\frac{C_{n+m+s-2}^{(n+m+s)}C_{n+s}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r)C_{n+m+3}^{(n+m+s)}} - \frac{C_{n+s-2}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r-2)}\right\}\frac{K_1\tau_1}{R_{m+s+1}} - \frac{K_1^2\tau_{m+s}}{\prod_{r=1}^{m}(n+s+r-2)R_{m+s+1}}\ \forall\, m+s=3.$$

Thus from (1.15), replacing ϕ_n with $\hat{\phi}_n$ we have the following expression for ε^*

$$\varepsilon^* = \left\{\frac{C_{n+m+s-2}^{(n+m+s)}C_{n+s}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r)C_{n+m+3}^{(n+m+3)}} - \frac{C_{n+s-2}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r-2)}\right\}\frac{\tau_1}{R_{m+s+1}} - \frac{\text{K}_1\tau_{m+s}}{\prod_{r=1}^{m}(n+s+r-2)R_{m+s+1}}\ \forall\, m+s=3$$

Numerical Examples

Here, we consider the application of the tau system and general error estimation formula obtained for the class of ordinary differential equations characterized by m +s = 3 to some examples. The exact error is defined as

$$\varepsilon *= \max_{0\le x\le 1}\{|y(x_k) - y_n(x_k)|\},\ \ 0 \le x \le 1,\ \ for\ k = 0(1)100,\ \{x_k\} = \{0.01k\}$$

Lapai Journal of Science and Technology, vol. 2, No.2 (2014)

Example 1

A First Order Homogeneous Linear Variable Coefficient Problem

$$y'(x) - x^2y(x) = 0 \qquad (4.2)$$

With the exact solution $y(x) = \exp(\frac{x^3}{3}),\ 0 \le x \le 1$ $y(0) = 1$ in this case m = 1, = 2.

Computed results are shown in table 4.1 below

Table 1: Error and Error Estimates for Example 1

Error Degree(n)	2	3	4	6
ε	$1.46.\times 10^{-6}$	2.18×10^{-7}	1.48×10^{-10}	4.13×10^{-12}
ε^*	2.40×10^{-6}	5.23×10^{-7}	4.60×10^{-10}	2.34×10^{-11}

Example 2

Third Order Non-Homogeneous Constant Coefficient Problem

$$Ly(x) = y'''(x) + 8y'(x) - 6x^2 + 9x + 2 \qquad (4.2a)$$

$$y(0) = 0, \quad y'(0) = 0,\ y''(0) = \frac{11}{128} \qquad (4.2b)$$

With the exact solution $y(x) = \frac{11x^2}{256} + \frac{7x^3}{32} - \frac{x^4}{16},\ 0 \le x \le 1$. For this problem m = 3 ,s = 0 .See the numerical examples in table 4.2 below

Table 2: Error and Error Estimates for Example 2

Error Degree(n)	2	3	4	6
ε	9.08×10^{-5}	6.03×10^{-5}	4.94×10^{-6}	2.84×10^{-7}
ε^*	7.10×10^{-3}	3.31×10^{-5}	4.16×10^{-6}	8.00×10^{-8}

CONCLUSION

The Lanzcos (1956) error estimation procedure is applicable to the class of first order linear ordinary differential equations with polynomial coefficients and whose solutions are given in the interval [0, 1]. The procedure is restricted to first order differential system which is good enough. The method of Fox can handle similar problems of order one and of higher orders other than one but is not general in scope of its application.The practical error estimation procedure of Onumanyi and Ortiz (1982) gives very accurate estimates. This is due to the idea of the Tau method.Though, the procedure is not economical considering the cost of computing because it involves the inversion of a matrix of dimension of at least (m + s) dimension.

The present error estimation technique shows a remarkable improvement over these works done on the subject of error analysis of the Tau method as it leads to error estimation formula with wider scope of application. Also, the estimate proposed here does not involve any iteration for linear problems nor matrix inversion. This is desirable in handling non-linear problems, where s, the number of overdetermination of the equation being considered, depend on n, the degree of the Tau approximant being sought and for large value of n, (m + s) then becomes very large. All these features are desirable and render the error estimation technique attractive for use with software design for the Tau method.

The results obtained in the present work demonstrate the closeness between the exact error of the tau method, thus error estimate of the tau method is effective and reliable. Our results are validated with numerical examples.

REFERENCES

Ma’ali, A. I. (2012): A Computational Error Estimation of the Integral Formulation of the Tau Method for some class of Ordinary Differential Equations. Doctoral Thesis (Unpublished), University of Illorin, Illorin.

Adeniyi, R. B. and Aliyu, A. I. M. (2012a). An Error Estimation of the Tau method for some class of ordinary differential equations. *IOSR Journal of Mathematics*, 2(1): 32-40. Retrieved from http://iosrjournals.

Adeniyi, R. B. and Aliyu, A. I. M. (2012b). On the integrated formulation of the Tau method involving at most two Tau parameters, for IVPS in odes *IOSR Journal of Mathematics*, 2(1): 23-31. Retrieved from http://iosrjournals.

Adeniyi, R. B. and Ma'ali, A. I. (2008). On the Tau method for a class of Non-overdetermined second differential equation. *Journal of the Nigerian Association of Mathematical Physics*, 12: 387-398.

Ma'ali, .A. I. (2008). On the Tau method for solution of a class of second order differential equations. M.Sc. Thesis, University of Ilorin, Ilorin.

Adeniyi, R. B. (2008). An improved error estimation of the Tau method for Boundary Values problems. *Research Journal of Applied Sciences*, 3(6): 456-464.

Adeniyi, R. B. and Erebholo, F. O. (2007). An error estimation of a numerical integration scheme for certain initial boundary value problem in differential equation. *Journal of Nigerian Mathematical Society*. 26: 99-109.

Adeniyi, R. B. (1991). On the Tau method for numerical solution of ordinary differential equations, Doctoral thesis, University of Ilorin, Nigeria.

Adeniyi, R. B. and Onumanyi, P. (1991). Error Estimation in the numerical solution of the ordinary differential equation with Tau method. *Computer mathematics Application*, 21(9): 19-27.

Adeniyi, R. B., Onumanyi, P. and Taiwo, .O. A. (1990). A computational error estimation of the Tau method for non-linear ordinary differential equation. *Journal of Nigerian Mathematics Society*, 9: 21-32.

Fox, I. and Parker, I. B. (1981). *Chebyshev polynomials in numerical analysis*. Oxford: Oxford University Press. P 8

Lanczos, C. (1938). Trigonometric interpolation of empirical and analytic functions, *Journal of Mathematics and Physics*, 17: 123-199

Lanczos, C. (1956). *Applied Analysis*. New Jersey: Prentice Hall. P. 67

Ortiz, E. L. (1974). Canonical polynomials in the Lanzcos Tau method, studies in numerical analysis. New York: Academic Press. P. 56-67

Ortiz, E. L. (1975). Step-by-step Tau Method, Piece-wise Polynomial Approximations. *Computer and Mathematics with Applications*, 1:381-392.