Robustness of Split-plot Central Composite Designs in the Presence of a Single Missing Observation.

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ABSTRACT

In this work, we present a Minimaxloss criterion, based on *D*-efficiency, and which accounts for the within-whole plot correlation among the observations, for constructing split-plot response surface designs that are robust to missing a single observation of the various design points. We then develop robust split-plot designs, which are relatively insensitive to a single missing observation. It was observed that the criterion is robust to changes in the degree of correlation among the observations but varies with changes in the number of whole-plot and subplot factors for a given *k* –factor design.

(Keywords: Split-plot response surface designs, robust designs, Minimaxloss criterion, within-whole plot correlation, missing observation)

INTRODUCTION

Response Surface Methodology (RSM) is an area of experimental design which consists of a group of mathematical and statistical techniques useful for developing, improving, and optimizing processes (Myers *et al*, 2009).

The formal development of response surface methodology started with the work of Box and Wilson (1951). Many books and papers discussing RSM have been published since the appearance of this seminal paper. The articles by Hill and Hunter (1966), Myers et al. (1989) and Mead and Pike (1975) provide a broad review of RSM. The books by Khuri and Cornell (1996), Box and Draper (2007), and Myers et al (2009), give a comprehensive coverage of the various techniques used in RSM. In practice, the form of the relationship between the two types of design variables is unknown but can be approximated, within the experimental region, by a first -order polynomial model of the form

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{e} \tag{1}$$

Where **y** is the response vector, $\boldsymbol{X}$ is the *n x p* model matrix, $\boldsymbol{\beta}$ is the *p x 1* vector of coefficients, and $\boldsymbol{e}$ is the *n x 1* vector of random errors.

RSM is sequential in nature. First, a first-order model is fit to the data from a 2^k design, the model is examined for *lack of fit* and when this is exhibited (by existence of surface curvature), axial runs are added to allow the quadratic terms to be incorporated into the model to give a 2nd - order model.

There are many designs available for fitting a second-order model. The most popular ones include the 3^k factorial designs and their fractional replicates, the central composite designs (CCD) introduced by Box and Wilson (1951), and the Box-Behnken designs (BBD) introduced by Box and Behnken (1960).

The CCD consists of factors with five levels that involve three categories. They are:

i. a complete (or a fraction of) 2^k factorial design with factor levels coded as -1, 1 (called the factorial portion),

ii. an axial portion consisting of 2*k* points arranged so that two points are chosen on the coordinate axis of each control variable at a distance of α from the design center,

iii. n_0 center points.

Thus the total number of points in a CCD is $n = 2^k + 2k + n_0$. The second-order response surface model for these designs is:

$$y_u = \beta_0 + \sum_{i=1}^{k} \beta_i x_{iu} + \sum_{i=1}^{k} \beta_{ii} x_{iu}^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{ij} x_{iu} x_{ju} \quad \textbf{(2)}$$

Where y is the response variable, X is the input variable, β is a model coefficient, and e is a random error component.

MISSING OBSERVATIONS IN DESIGNED EXPERIMENTS

In an experimental work, situation often arises where some observations are lost or unavailable due to some accidents or cost constraints. Missing observations can occur as a result of many causes during the conduct of an experiment. An observation may be lost, animals can invade and destroy some experimental units, floods or fires can occur and damage a part of the experiment, and on some occasions workers have been known to unintentionally leave out some of the experimental units when setting up the experiment. Missing observations can create a big problem by making the results of a response surface experiment quite misleading, thereby adversely affecting the inference. Unavailability of some observations destroys some useful properties of the design, such as orthogonality, rotatability, and optimality. The design may break down as a result of missing observations (Herzberg and Andrews (1976) and Andrews and Herzberg (1979)).

The data with missing observations may be handled by dropping the corresponding rows of the model matrix and then proceed with the analysis of the remaining (reduced) data using least squares procedure. However, dropping the rows of X amounts to changing the design structure and this adversely affects the useful design properties given above. Or, the remaining data may be handled by obtaining estimates for the missing observations using the techniques provided by some authors (e.g., Allan and Wishart (1930a), Yates (1933), Anderson (1946), Cochran and Cox, (1957)), etc., for computing missing plot values, substitute these estimates in to the data and then proceed with the analysis.

All these techniques only make the analysis of the remaining data as simple as possible but do not guard against the loss incurred by the experiment when some observations are missing. To minimize the effects of missing observations, we require designs which are robust to missing observations. In such robust designs, the parameters of the assumed model can be estimated without much loss of efficiency.

THE MINIMAXLOSS CRITERION

Central Composite Designs with different properties can be developed by taking different values of α, i.e. distance of axial points from the center of the design. Box (1954) developed orthogonal CCD. Box and Hunter (1957) developed rotatable designs. Box and Draper (1959) discussed designs robust to inadequate model. Box and Draper (1975) studied designs robust to outliers, which are referred here as outlier robust designs. Designs robust to missing observations with different probability of missing for different observations are studied by Herzberg and Andrews (1975, 1976) and Andrews and Herzberg (1979).

Akhtar and Prescott (1986) introduced a loss criterion, based on *D* – efficiency, for completely randomized response surface designs robust to missing observations, which minimizes the maximum loss due to missing observations. The loss is the relative reduction in the information matrix of these designs when one or more observations are missing. They defined the loss for the *i*th missing design point as:

$$L_i = 1 - R_i = h_{ii} \quad \textbf{(3)}$$

The loss L_i is a relative measure of efficiency with $0 \le L_i \le 1$. A small value of L_i indicates a low reduction in the determinant of the information matrix and, in this sense, less loss of information.

Akhtar and Prescott (1986) used this criterion to develop standard central composite designs robust to any single or any pair of missing observations. They showed that when two observations are missing, say, the *i*th and *j*th observations, then,

$$l_{ij} = 1 - \left((1 - h_{ii})(1 - h_{jj}) - h_{ij}^2\right) \quad \textbf{(4)}$$

where h_{ii}, h_{jj}, and h_{ij} are corresponding elements of the design's 'hat' matrix,

$$H = X(X'X)^{-1}X' \quad \textbf{(5)}$$

When the *i*th, *j*th and *l*th observations are missing, then,

$$l_{ijl} = 1 - [(1 - h_{ii})(1 - h_{jj})(1 - h_{ll}) - h_{il}^2(1 - h_{jj}) - h_{ij}^2(1 - h_{ll}) - h_{jl}^2(1 - h_{ii}) - 2h_{ij}h_{il}h_{jl}] \quad \textbf{(6)}$$

Akhtar (2001) investigates two missing values in three different configurations of factorial and axial parts of completely randomized CCDs with five factors and construct five-factor CCDs that are robust to a pair of missing observations under the Minimaxloss criterion. The configurations are (i) designs with half replicate of factorial part and complete replicate of axial part (1/2 F+A), (ii) designs with one replication of factorial and axial parts (F+A), and (iii) designs with one replication of factorial part and two replications of axial part (F+2A).

Akram (2002) studies the robustness of completely randomized central composite designs with $2 \leq k \leq 6$, to all possible combinations of ***three missing*** observations with three different configurations, and developed CCDs that are robust to any three missing observations using the minimaxloss criterion.

Ahmad and Gilmour (2010) study the robustness of subset response surface designs to a single missing observation and developed such designs that are robust to one missing observation. The subset designs are a wide class of three-level response surface designs introduced by Gilmour (2006).

Ahmad (2011) studies and constructs different types of second-order response surface designs which are more robust to missing data than the competitive designs of the similar structure in the literature using the minimaxloss criterion.

Martin *et al* (2013) studies the robustness of three-level response surface designs against missing data under the following criterion:- *the maximum number of observations that can be missing from a design and still allow the estimation of the given model with a high probability*.

All these studies are on response surface experiments conducted in completely randomized (CRD) mode, which involve a single experimental unit, one level of randomization, single error structure, and independent observations.

RESPONSE SURFACE DESIGNS WITH RESTRICTED RANDOMIZATION

In most industrial experiments complete randomization is not achievable due to the presence of hard-to-change (HTC) factors. Instead, the experimenter approaches these experiments in an appropriate manner that restricts the randomization and that leads to a ***split-plot*** structure. In a split-plot design, the experimental runs are performed in groups, where, in a group, the levels of the HTC factors are not reset.

Non-resetting of factors creates a dependence among the runs in one group, and this results in clusters of correlated errors and responses. In fact, it has been claimed long ago that all industrial experiments are split-plot experiments and this has been confirmed by recent works that many experiments previously thought to be CRD experiments also exhibit a split-plot structure. For details, see Daniel (1976), Anbari and Lucas (1994), Ganju and Lucas (1997, 1999, 2005), Ju and Lucas (2002), and Webb *et al* (2004). This surprising result has motivated a great deal of pioneering work in the design and analysis of split-plot experiments today. These designs consist of two different randomization procedures (whole plot and subplot) for the experimental runs, which lead to two error terms – the whole plot error term (σ_γ^2) and the subplot error term (σ_ε^2). Their performance depends, therefore, on the relative magnitude of the two variance components.

Letsinger *et al*. (1996) is the first paper to investigate the efficiency of various second-order response surface designs when run as a split-plot experiment. Then, Draper and John (1998) discussed modifications of central composite designs and Box-Behnken designs to be run in a split-plot format. A sequential strategy for designing multistratum designs, special cases of which are split-plot designs, was presented by Trinca and Gilmour (2001).

The optimal design of first- and second-order split-plot experiments later received attention by Goos and Vandebroek (2001b, 2003a, 2004). Goos (2002) provided a thorough development of the D-optimal split-plot design approach including implementation strategies. The optimal design of split-plot experiments for spherical design regions receives special attention in Mee (2006).

Vining, Kowalski and Montgomery (VKM) (2005) modify the standard CCD to accommodate a split-plot structure and illustrate the construction of split-plot response surface designs that are based on the original CCD of Box and Wilson (1951) and BBD of Box and Behnken (1960). The authors then develop designs that achieve the equivalence of OLS and GLS estimates, and establish the general conditions for equivalent estimation designs. VKM (2005) also show that these designs provide estimates of pure-error at both the whole-plot and subplot levels, thereby producing a model-independent estimate of the variance-covariance matrix, ***V***.

Kowalski *et al.* (2006) modifies the VKM (2005) CCD to allow the estimation of separate models for the characteristic's mean and variances under a split-plot structure and note that the OLS -GLS equivalence will no longer hold for these designs, and discuss how to estimate the terms in both models.

As we have seen, missing observations in response surface designs conducted within a completely randomized (CRD) mode, especially the central composite designs (CCD), are well investigated since the development of Minimaxloss criterion presented by Akhtar and Prescott [1986]. But the same is not true for such designs conducted within a split-plot structure for which the performance depends on the unknown variance components.

This work therefore extends the form of the minimaxloss criterion of Akhtar and Prescott (1986) for constructing completely randomized response surface designs robust to missing observations, to the response surface designs conducted within a split-plot structure.

The impact of missing a single observation of the different design points on the robustness of these designs is investigated and robust designs were constructed under each of the following configurations:- (i) single replicate designs and (ii) designs with half factorial replicate and full axial replicate.

MODEL AND NOTATIONS

In a split-plot design, observations within a whole plot are correlated and those from different whole plots are independent. To account for such correlation among the observations, the generalized least squares (GLS) model below is required to obtain estimates of the model parameters:

$$\boldsymbol{y} = \boldsymbol{X\beta} + Z\gamma + \varepsilon \qquad \textbf{(7)}$$

Where $\boldsymbol{y}$ is the *N x 1* vector of responses, $\boldsymbol{X}$ is the *N x p* model matrix, $\boldsymbol{\beta}$ is the *p* x 1 vector of coefficients, Z is an *N x b* incidence matrix assigning observations to each of the *b* whole plots; γ is the *N x 1* vector of random whole-plot errors, ε is the *N x 1* vector of random subplot errors. It is assumed that $\gamma_i \sim N(0, \sigma_\gamma^2), \quad \varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2), \; cov(\gamma_i, \varepsilon_{ij}) = 0.$

The variance - covariance matrix for the observation vector **y** is

$$Var(y) = V = \sigma_\varepsilon^2 I_n + \sigma_\gamma^2 ZZ' \\ = \sigma_\varepsilon^2 (I_n + dZZ') \qquad \textbf{(8)}$$

where $d = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$.

The matrix $\boldsymbol{ZZ}$' is a block diagonal matrix with diagonal matrices of $J_{n1}, J_{n2}, \ldots, J_{nz}$, where J_{ni} is an n_i x n_i matrix of 1's, and $\sigma_\gamma^2 / \sigma_\varepsilon^2$ denotes the ratio of the two variance components.

MATERIALS AND METHODS

Robustness of split-plot central composite designs with $2 \leq k \leq 6$ factors were investigated in the presence of a single missing observation under each of the following configurations using the proposed minimaxloss criterion: (i) designs with single replication of factorial and axial parts (F+A) and (ii) designs with half replicate of factorial part and complete replicate of axial part (1/2 F+A).

Under the first configuration (i.e., F+A), six split-plot central composite designs (SP-CCDs) comprising of one 2-factor design (D(1,1)), two 3-factor designs differing in numbers of whole plot and subplot variables (i.e., D(1,2) and D(2,1)), and three 4-factor designs also with different numbers of whole plot and subplot variables (i.e., D(1,3), D(3,1), and D(2,2)) were considered. Under the second configuration (i.e., 1/2F+A), three split-plot designs were considered consisting of two 5-factor designs differing in numbers of whole plot and subplot variables (D(1,4) and D(2,3)), and one 6-factor design (D(2,4)).

Now for split-plot response surface designs, the generalized least squares (GLS) estimates are

$$\hat{\beta}_{GLS} = (X'V^{-1}X)^{-1}X'V^{-1}y$$

$$Var(\hat{\beta}_{GLS}) = (X'V^{-1}X)^{-1}$$

$$\hat{y} = X(X'V^{-1}X)^{-1}X'V^{-1}y = Hy$$

$$Var(\hat{y}) = [X(X'V^{-1}X)^{-1}X'] = VH \qquad \textbf{(9)}$$

where ***X*** is the model matrix, **y** is the vector of responses and *H* is the 'hat' matrix for the split-plot CCD.

We denote the determinant of the information matrix for the complete split-plot central composite design by:

$$d = |(X'V^{-1}X)| \qquad \textbf{(10)}$$

Under *D*-optimality, we maximize (10), and for these designs our computations have shown that (10) is an increasing function of α and is maximum at $\alpha = \infty$, where α is the axial point distance from the design center.

We denote the determinant of the information matrix for the reduced design by:

$$d_r = |(X_r'V_r^{-1}X_r)| \qquad \textbf{(11)}$$

If there are two missing observations *u* and *v*, (10) is reduced to:

$$d = |(X'V^{-1}X)|_{uv}. \qquad \textbf{(12)}$$

We want this reduction to be as small as possible. Therefore, since the loss L_i is a relative measure of *D*-efficiency, we define the minimaxloss criterion due to the *u*th missing point in a split-plot CCD as:

$$l_u = \frac{|X'V^{-1}X| - |X'V^{-1}X|_u}{|X'V^{-1}X|} = 1 - \frac{|X'V^{-1}X|_u}{|X'V^{-1}X|},\ u = 1, 2, \ldots \qquad \textbf{(13)}$$

Where $|X'V^{-1}X|_u$ is the determinant of the reduced information matrix due to the missing u[th] point. The values of l_u lie between zero and one, that is, $0 \le l_u \le 1$. If $l_u = 1$, then $1 - \frac{d_r}{d} = 1$ and this shows that the determinant of the reduced information matrix is zero and the design may break down (Herzberg and Andrews, 1976).

The criterion in (13) is used in this work to construct split-plot central composite designs with $2 \le k \le 6$ factors and at different values of $(\alpha = \beta)$, that are robust to a *single* **missing** observation of the different design points under the two different configurations mentioned above, for various degrees of correlation (***d***).

Now, a split-plot CCD consists of four different kinds of design points. These include the factorial design points (n_f), the whole-plot axial points (n_α), the subplot axial points (n_β), and the central points (n_c). Each of these points has different effects on the design when their corresponding observations or combinations of observations are missing. In this work we investigate the robustness of split-plot central composite designs (CCD) with *w* whole plot variables (*w* = 1, 2) and *s* subplot variables (*s* = 2, 3, 4), which we denote as D(*w,s*), for a second order model with the fixed effect in the form:

$$f(z,x) = \beta_0 + \sum_{i=1}^{w}\beta_i z_i + \sum_{i=1}^{w-1}\sum_{j=i+1}^{w}\beta_{ij}z_i z_j + \sum_{i=1}^{w}\beta_{ii}z_i^2 + \sum_{i=1}^{s}\theta_i x_i + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\theta_{ij}x_i x_j + \sum_{i=1}^{w}\sum_{j=1}^{s}\gamma_{ij}z_i x_j + \sum_{i=1}^{s}\theta_{ii}x_i^2$$

(14)

Where z is a whole-plot factor, x is a subplot factor, the $\beta's$ are the regression coefficients at the whole-plot levels, $\theta's$ and $\gamma's$ are the regression coefficients at the subplot levels.

In all the designs considered in this work, the missing observation is only at the subplot level, and the whole -plot axial point distance equals the subplot axial point distance from the design center (i.e., α = β). Then a split-plot response surface design under a given configuration is regarded as a **minimaloss design** robust to a single missing observation**,** at a given value of $(\alpha = \beta)$, when the loss due to a missing factorial point equals the loss due to a missing subplot axial point, and such a loss is also the **minimum** among all the **maximum** losses for the different types of design points.

RESULTS AND DISCUSSIONS

Tables of maximum losses due to missing a single observation of each of these design points were constructed at different values of $(\alpha = \beta)$ for each of the different $D(w, s)$ split-plot central composite designs with $2 \leq k \leq 6$ factors, n_f factorial, n_α whole plot axial, n_β subplot axial and n_c center points, under each of the two configurations. These losses are denoted as L_c, L_α, L_f, and L_β corresponding to missing a center, a whole plot axial, a factorial, and a subplot axial points, respectively.

Our computations of losses have shown that, for each of these designs, the loss due to missing an observation is robust to changes in the ratio of the two variance components but varies with changes in the number of whole-plot and subplot factors for a given *k* –factor design. It was also observed that the loss corresponding to missing a center point, (L_c), and that corresponding to missing an axial point (L_α), are each less than L_f and L_β for the whole range of α, and also that L_c continues to increase up to the point $\alpha = \sqrt{k}$, and then decreases as α increases beyond this point. It can also be seen from each of the tables that the loss L_f decreases gradually with increasing value of α, whereas L_β has an increasing trend with increasing α.

DESIGNS WITH SINGLE REPLICATION OF FACTORIAL AND AXIAL PARTS (F+A)

Under this configuration, the losses corresponding to missing a single observation of each of the different categories of design points are given in Tables 1 – 6, respectively, for the split-plot central composite designs D(1,1), D(2,1), D(1,2), D(1,3), D(3,1), and D(2,2). These losses are also plotted against the various α values and given in the corresponding Figures 1-6 as shown below for each of the designs.

Comparing the different losses, it can be seen from each of the tables that there is a point (α) where the values of L_f and L_β coincide, as can also be seen visually from the corresponding figures. That is, at this point, the maximum loss is minimized. For the design D(1,1) in Table 1, this point is at $\alpha = 1.5946$, and thus the D(1,1) split-plot CCD with n_f = 4, n_α = 4, n_β = 2, n_c = 2, and $\boldsymbol{\alpha = \beta = 1.5946}$ is a minimaxloss design robust to a single missing observation.

From Table 2, we see that the three-factor split-plot CCD with,

$$w = 2,\ s = 1,\ n_f = 8,\ n_\alpha = 8,\ n_\beta = 2,\ n_c = 2 \text{ and } \boldsymbol{\alpha = \beta = 2.032}$$

is a minimaxloss design robust to a single missing observation. From Table 3, we can see that the three-factor split-plot CCD with,

$$w = 1,\ s = 2,\ n_f = 8,\ n_\alpha = 8,\ n_\beta = 4,\ n_c = 4 \text{ and } \boldsymbol{\alpha = \beta = 1.912}$$

is a minimaxloss design robust to a single missing observation. From Table 4, we observed that the values of L_f and L_β coincide at the point $\boldsymbol{\alpha = \beta = 1.856}$ for the design D(1,3)**.** Thus the four-factor split-plot CCD with,

$$w = 1, s = 3,\ n_f = 16,\ n_\alpha = 16, n_\beta = 6,\ n_c = 8, \text{ and } \boldsymbol{\alpha = \beta = 1.856}$$

is a minimaxloss design robust to a single missing observation. We can observe from Table 5 that for the design D(3,1), the values of L_f and L_β coincide at the point where $\boldsymbol{\alpha = \beta = 2.2007}$.

Thus, the four-factor split-plot CCD with,
$w = 3,\ s = 1,\ n_f = 16,\ n_\alpha = 12,\ n_\beta = 2,\ n_c = 2,$ and $\alpha = \beta = 2.2007$

is a minimaxloss design robust to a single missing observation. We observe from TABLE 3.6 that the four-factor split-plot CCD with,

$w = 2,\ s = 2,\ n_f = 16,\ n_\alpha = 16,\ n_\beta = 4,\ n_c = 2,$ and $\boldsymbol{\alpha = \beta = 1.875}$

is a minimaxloss design robust to a single missing observation.

These points of minimum can also be seen visually from each of the corresponding Figures 1-6 where the loss curves L_f and L_β intersect at the value of $(\alpha = \beta)$.

We can see from Figures 1, 2, 5, and 6 that the loss L_c makes a bell-shaped curve when plotted against α, attaining it's maximum at $\alpha = \sqrt{k}$, while L_α increases gradually with increasing α. Figure 3 shows that L_c slightly makes a bell-shaped curve when plotted against α, attaining its maximum at $\alpha = \sqrt{3}$, while L_α is almost horizontal for the whole range of α. In Figure 4, we observed that the loss curves L_c and L_α maintain almost the same value for the whole range of α.

DESIGNS WITH HALF REPLICATION OF FACTORIAL PART AND COMPLETE REPLICATION OF AXIAL PART (1/2F+A)

Under this configuration, the losses corresponding to missing a single observation of each of the different categories of design points are given in Tables 7-9, respectively, for the split-plot designs D(1,4), D(2,3) and D(2,4). These losses are also plotted against the various α values and given in the corresponding Figures 7-9 as shown below for each of the designs.

Our computations of losses have shown that, for each of these designs also, the loss corresponding to missing a center point, (L_c), and that corresponding to missing an axial point (L_α), are each less than L_f and L_β for the whole range of α, and also that L_c continues to increase up to the point $\alpha = \sqrt{k}$, and then decreases as α increases beyond this point. It can also be seen from each of the tables that the loss L_f decreases gradually with increasing value of α, whereas L_β has an increasing trend with increasing α.

From the losses in Table 7 for the D(1,4) design, we observed that the maximum loss $(L_\beta = 0.8047)$ is minimum when $L_f = L_\beta = 0.8047$ for $\alpha = \beta = 3.8947$ as indicated in the last column of the table. Thus, the five-factor D(1,4) split-plot CCD with half replication of factorial part and complete replication of axial part, and with,

$n_f = 16,\ n_\alpha = 16,\ n_\beta = 8,\ n_c = 8,$ and $\alpha = \beta = 3.8947$

is a minimaxloss design robust to a single missing observation. From Table 8, we observed that the maximum loss $(L_\beta = 0.8089)$ is minimum when $L_f = L_\beta = 0.8089$ for $\alpha = \beta = 4.0711$, as indicated in the last column of the table. Thus, the five-factor D(2,3) split-plot CCD with half replication of factorial part and complete replication of axial part, and with,

$n_f = 16,\ n_\alpha = 16,\ n_\beta = 6,\ n_c = 4,$ and $\alpha = \beta = 4.0711$

is a minimaxloss design robust to a single missing observation. We can observe from Table 9 that the maximum loss $(L_\beta = 0.5984)$ is minimum when $L_f = L_\beta = 0.5984$ for $\alpha = \beta = 2.626$, as indicated in the last column of the table. Thus, the six-factor D(2,4) split-plot CCD with half replication of factorial part and complete replication of axial part and with,

$n_f = 32,\ n_\alpha = 32,\ n_\beta = 8,\ n_c = 8,$ and $\alpha = \beta = 2.626$

is a minimaxloss design robust to a single missing observation.

These points of minimum can also be seen visually from each of the corresponding Figures 7-9 where the loss curves L_f and L_β intersect at the value of $(\alpha = \beta)$.

We can see from Figures 7 and 9 that the loss curves L_c and L_α almost maintain the same value for the whole range of α, while in Figure 8, the loss L_c makes a bell-shaped curve when plotted against α, attaining it's maximum at $\alpha = \sqrt{k}$, while L_α increases gradually with increasing α.

CONFIGURATION ONE

1. Two-Factor D(1,1) Split-Plot CCD

This design consists of one whole-plot factor (w =1) and one subplot factor (s = 1), n_f = 4 factorial points, n_α = 4 whole-plot axial points, n_β = 2 subplot axial points, and n_c = 2 center points. There are 6 whole plots each of size 2 and N = 12 total design points. The losses due to missing single observations of these different points are studied for range of α from 1.0 to 2.0, and are given in the table below. These losses are also plotted against α as given in Figure 1 below.

TABLE 3.1: Losses due to Single Missing Observations of Different Design Points for D(1,1) Split-plot CCD.

α	$\lvert X'V^{-1}X\rvert$ **for complete design**	**Loses due to missing a Center point (L_c)**	**Loses due to missing a factorial point (L_f)**	**Loses due to missing a whole-plot axial point (L_α)**	**Loses due to missing a subplot axial point (L_β)**	**Overall maxloss due to single missing observation**
1.0	2.2755E+2	0.1666	0.7916	0.1666	0.4166	0.7916
1.20	6.2718E+2	0.2224	0.7237	0.1768	0.4762	0.7237
$\sqrt{2}$ =1.414	2.1210E+3	0.2499	0.6577	0.1904	0.5534	0.6577
1.50	3.6188E+3	0.2461	0.6376	0.1962	0.5859	0.6376
1.5946	6.6133E+3	0.2353	0.6200**	0.2022	0.6200**	0.6200**
1.70	1.2991E+4	0.2193	0.6048	0.2082	0.6546	0.6546
1.90	4.5199E+4	0.1889	0.5840	0.2169	0.7090	0.7090
2.0	8.1920E+4	0.1759	0.5759	0.2203	0.7314	0.7314

**Minimaxloss due to one missing observation.

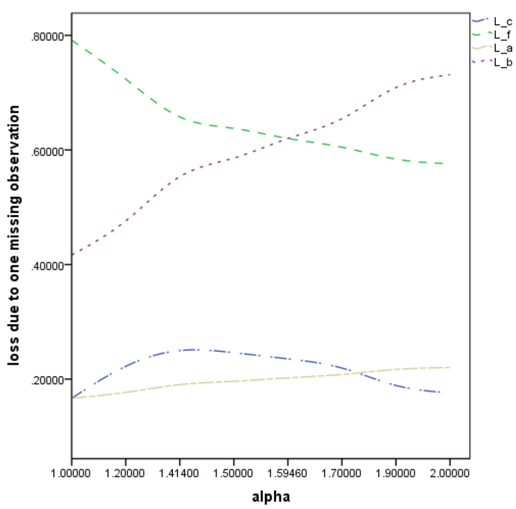

Figure 1: Loss curves due to one missing observation for a D(1,1) split-plot CCD with n_f = 4, n_α = 4, n_β = 2, and n_c = 2.

2. Three-Factor D(2,1) Split-plot CCD.

A three-factor split-plot CCD in two whole plot and one subplot variables and with single replication of factorial and axial parts consists of n_f = 8 factorial points, n_α = 8 whole-plot axial points, n_β = 2 subplot axial points, and n_c = 2 center points. There are 10 whole plots each of size 2 and N = 20 total design points. The losses due to missing single observations of these different points are studied for range of α from 1.0 to 3.0, and are given in the table below. These losses are also plotted against α as given in Figure 2 below.

Table 2: Losses due to Single Missing Observations of Different Design Points for D(2,1) Split-plot CCD.

α	$\lvert X'V^{-1}X \rvert$ for complete design	Loses due to missing a Center point (L_c)	Loses due to missing a factorial point (L_f)	Loses due to missing a whole-plot axial point (L_α)	Loses due to missing a subplot axial point (L_β)	Overall maxloss due to single missing observation
1.0	5.3939E+5	0.0900	0.7308	0.1566	0.3600	0.7308
1.50	2.1241E+7	0.2132	0.6581	0.1748	0.4546	0.6581
1.7320	1.1549E+8	0.2499	0.6240	0.1852	0.5125	0.6240
1.85	2.9328E+8	0.2412	0.6113	0.1916	0.5468	0.6113
2.0	9.8978E+8	0.2142	0.5997	0.1994	0.5892	0.5997
2.032	1.2829E+9	0.2077	0.5977**	0.2009	0.5977**	0.5977**
2.2	4.8932E+9	0.1747	0.5888	0.2078	0.6383	0.6383
2.5	4.5584E+10	0.1317	0.5761	0.2168	0.6961	0.6961
3.0	1.1529E+12	0.0954	0.5586	0.2261	0.7649	0.7649

**Minimaxloss due to one missing observation

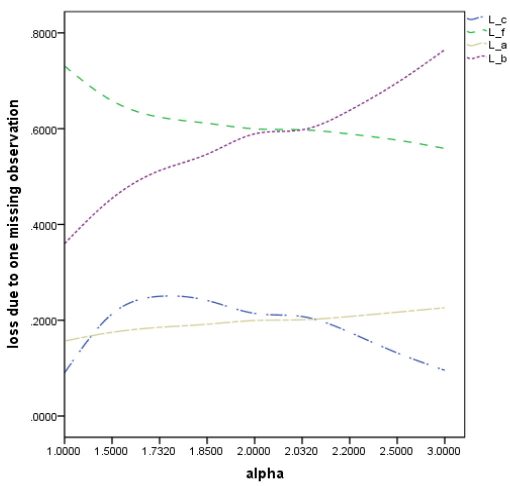

Figure 2: Loss curves due to one missing observation for a 3-factor split-plot CCD with two whole plot and one subplot variables, n_f = 8, n_α = 8, n_β = 2, and n_c = 2.

3. Three-Factor D(1,2) Split-Plot CCD.

A three-factor split-plot CCD in one whole plot and two subplot variables and with one replication of factorial and axial parts consists of n_f = 8 factorial points, n_α = 8 whole-plot axial points, n_β = 4 subplot axial points, and n_c = 4 center points. There are 6 whole plots each of size 4, and N = 24 total design points. The losses due to missing single observations of these different points are studied for range of α from 1.0 to 3.0, and are given in the table below. These losses are also plotted against α as given in Figure 3 below.

Table 3: Losses due to Single Missing Observations of Different Design Points for D(1,2) Split-plot CCD.

α	$\lvert X'V^{-1}X \rvert$ for complete design	Loses due to missing a Center point (L_c)	Loses due to missing a factorial point (L_f)	Loses due to missing a whole- plot axial point (L_α)	Loses due to missing a subplot axial point (L_β)	Overall maxloss due to single missing observation
1.0	0.3019E+7	0.0347	0.7638	0.0451	0.4722	0.7638
1.20	0.1660E+8	0.0434	0.7385	0.0477	0.5090	0.7385
1.50	0.1658E+9	0.0578	0.6991	0.0508	0.5670	0.6991
1.73205	0.9588E+9	0.0624	0.6707	0.0529	0.6150	0.6707
1.77243	0.1305E+10	0.0623	0.6662	0.0533	0.6234	0.6662
1.90	0.3465E+10	0.0604	0.6531	0.0543	0.6495	0.6531
1.912	0.3798E+10	0.0601	0.6520**	0.0544	0.6520**	0.6520**
2.0	0.7421E+10	0.0578	0.6439	0.0550	0.6689	0.6689
2.2	0.3293E+11	0.0517	0.6280	0.0563	0.7044	0.7044
2.5	0.2698E+12	0.0434	0.6086	0.0577	0.7488	0.7488
3.0	0.6027E+13	0.0347	0.5836	0.0591	0.8046	0.8046

**Minimaxloss due to one missing observation.

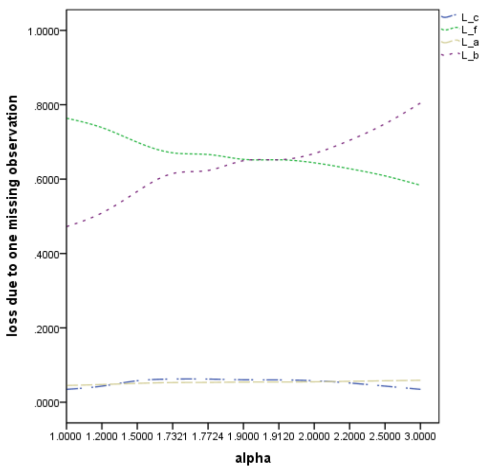

Figure 3: Loss curves due to one missing observation for a 3-factor split-plot CCD with one whole plot and two subplot variables, n_f = 8, n_α = 8, n_β = 4, and n_c = 4.

4. Four-Factor D(1,3) Split-plot CCD

A four-factor split-plot CCD in one whole-plot and three subplot variables, and with single replication of factorial and axial parts consists of n_f = 16 factorial points, n_α = 16 whole-plot axial points, n_β = 6 subplot axial points and n_c = 8 center points. The design consists of N = 46 total design points and 6 whole plots. The losses due to missing single observations of these different points are studied for range of α from 1.0 to 3.0, and are given in the table below. These losses are also plotted against α as given in Figure 4 below.

Table 4: Losses due to Single Missing Observations of Different Design Points for D(1,3) split-plot CCD.

α	$\lvert X'V^{-1}X \rvert$ for complete design	Losses due to missing a Center point (L_c)	Losses due to missing a factorial point (L_f)	Losses due to missing a whole-plot axial point (L_α)	Losses due to missing a subplot axial point (L_β)	Overall maxloss due to single missing observation
1.0	2.4294E+13	0.0083	0.6208	0.0115	0.4680	0.6208
1.20	2.7483E+14	0.0097	0.6111	0.0121	0.4907	0.6111
1.50	6.0377E+15	0.0125	0.5954	0.0129	0.5279	0.5954
1.75	6.0229E+16	0.0147	0.5817	0.0134	0.5613	0.5817
1.80	9.4006E+16	0.0150	0.5790	0.0135	0.5682	0.5790
1.856	1.5423E+17	0.0153	0.5761**	0.0136	0.5761**	0.5761**
2.0	5.4393E+17	0.0156	0.5683	0.0138	0.5965	0.5965
2.2	3.0477E+18	0.0151	0.5583	0.0141	0.6249	0.6249
2.5	3.7102E+19	0.0133	0.5447	0.0144	0.6649	0.6649
3.0	1.6952E+21	0.0104	0.5255	0.0147	0.7220	0.7220

**Minimaxloss due to one missing observation.

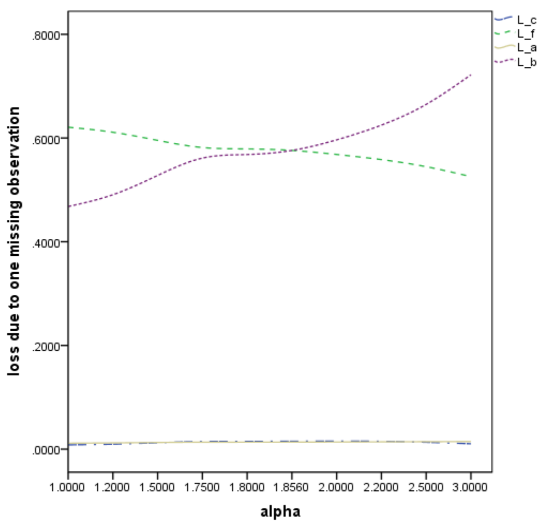

Figure 4: Loss curves due to one missing observation for a 4-factor split-plot CCD with one whole plot and three subplot variables, n_f = 16, n_α = 16, n_β = 6, and n_c = 8.

5. Four-Factor D(3,1) Split-plot CCD.

A four-factor split-plot CCD with three whole-plot and one subplot variables, and with one replication of factorial and axial parts consists of n_f = 16 factorial points, n_α = 12 whole-plot axial points, n_β = 2 subplot axial points and n_c = 2 center points. The losses due to missing single observation of these various design points are studied for values of α ranging from 1.0 to 3.0 and are given in the table below. These losses are also plotted against α as given in Figure 5 below.

Table 5: Losses due to Single Missing Observations of Different Design Points for D(3,1) Split-plot CCD.

α	$\lvert X'V^{-1}X \rvert$ for complete design	Losses due to missing a Center point (L_c)	Losses due to missing a factorial point (L_f)	Losses due to missing a whole- plot axial point (L_α)	Losses due to missing a subplot axial point (L_β)	Overall maxloss due to single missing observation
1.0	3.2469E+11	0.0564	0.5642	0.1446	0.3118	0.5642
1.25	5.2007E+12	0.0801	0.5510	0.1529	0.3439	0.5510
1.50	4.9939E+13	0.1264	0.5362	0.1599	0.3742	0.5362
1.75	3.6328E+14	0.2017	0.5190	0.1654	0.4026	0.5190
2.0	2.8115E+15	0.2500	0.5017	0.1736	0.4444	0.5017
2.2007	1.7478E+16	0.2210	0.4921**	0.1832	0.4921**	0.4921**
2.25	2.7763E+16	0.2086	0.4903	0.1856	0.5040	0.5040
2.5	2.8261E+17	0.1483	0.4832	0.1961	0.5588	0.5588
3.0	1.8541E+19	0.0838	0.4708	0.2098	0.6404	0.6404

**Minimaxloss due to one missing observation.

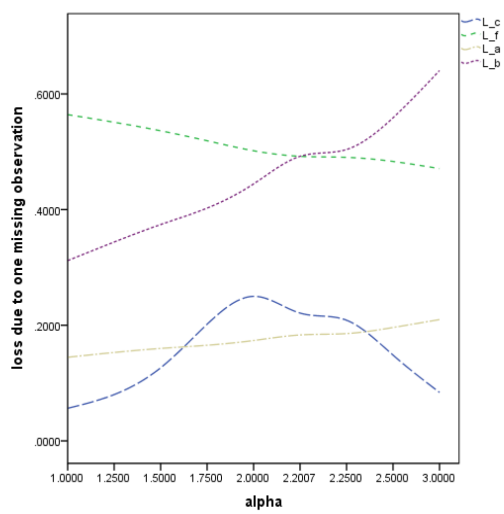

Figure 5: Loss curves due to one missing observation for a 4-factor split-plot CCD with three whole plot and one subplot variables, n_f = 16, n_α = 12, n_β = 2, and n_c = 2.

6. Four-Factor D(2,2) Split-plot CCD.

A four-factor split-plot CCD with two whole-plot and two subplot variables, and with one replication of factorial and axial parts consists of n_f = 16 factorial points, n_α = 16 whole-plot axial points, n_β = 4 subplot axial points and n_c = 2 center points. There are 10 whole plots and N = 38 total design points. The losses due to missing single observations of these different points are studied for range of α from 1.0 to 3.0, and are given in the table below. These losses are also plotted against α as given in Figure 6 below.

Table 6: Losses due to Single Missing Observations of Different Design Points for D(2,2) Split-plot CCD.

α	$\lvert X'V^{-1}X \rvert$ **for complete design**	**Losses due to missing a Center point (L_c)**	**Losses due to missing a factorial point (L_f)**	**Losses due to missing a whole- plot axial point (L_α)**	**Losses due to missing a subplot axial point (L_β)**	**Overall maxloss due to single missing observation**
1.0	2.1955E+12	0.0817	0.5813	0.0406	0.4288	0.5813
1.20	2.3611E+13	0.1030	0.5718	0.0431	0.4516	0.5718
1.50	4.5928E+14	0.1575	0.5565	0.0462	0.4870	0.5565
1.75	4.0784E+15	0.2186	0.5428	0.0482	0.5182	0.5428
1.85	9.6361E+15	0.2379	0.5373	0.0490	0.5322	0.5373
1.875	1.1965E+16	0.2415	0.5359**	0.0493	0.5359**	0.5359**
2.0	3.5860E+16	0.2500	0.5295	0.0503	0.5555	0.5555
2.2	2.1775E+17	0.2319	0.5205	0.0520	0.5892	0.5892
2.5	3.2216E+18	0.1771	0.5098	0.0541	0.6371	0.6371
3.0	1.9267E+20	0.1142	0.4953	0.0565	0.7013	0.7013

**Minimaxloss due to one missing observation.

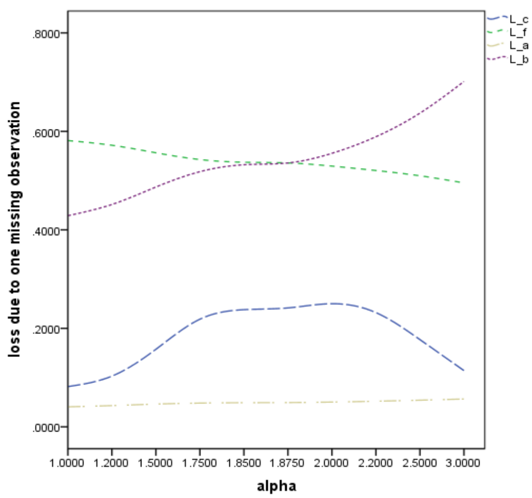

Figure 6: Loss curves due to one missing observation in a 4-factor split-plot CCD with 2 whole plot and 2 subplot variables, n_f = 16, n_α = 16, n_β = 4, and n_c = 2.

(2) Designs With Half Replication Of Factorial Part And Complete Replication Of Axial Part

7. Five-Factor D(1,4) Split-plot CCD

A five-factor split-plot CCD with one whole-plot and four subplot variables consists of n_f = 16 points from half replicate of factorial part with highest-order interaction as the defining contrast, n_α = 16 points of whole-plot axial part, n_β = 8 points of subplot axial part and n_c = 8 center points. There are 6 whole plots and N = 48 total design points. The losses due to missing single observations of these different points are studied for range of α from 1.0 to 4.5, and are given in the table below.

Table 7: Losses due to Single Missing Observations of Different Design Points for D(1,4) Split-plot CCD.

A	$\|X'V^{-1}X\|$ for complete design	Losses due to missing a Center point (L_c)	Losses due to missing a factorial point (L_f)	Losses due to missing a whole- plot axial point (L_α)	Losses due to missing a subplot axial point (L_β)	Overall maxloss due to single missing observation
1.0	7.6092E+19	0.0080	0.9647	0.0116	0.4924	0.9647
1.20	1.9017E+21	0.0090	0.9522	0.0122	0.5150	0.9522
1.50	1.1011E+23	0.0111	0.9321	0.0130	0.5515	0.9321
2.0	3.0447E+25	0.0150	0.8975	0.0139	0.6151	0.8975
2.5	4.5981E+27	0.0149	0.8658	0.0144	0.6775	0.8658
3.0	4.6119E+29	0.0122	0.8397	0.0147	0.7318	0.8397
3.424	1.5751E+31	0.0102	0.8215	0.0149	0.7699	0.8215
3.8947	5.3909E+32	0.0087	0.8047**	0.0151	0.8047**	0.8047**
4.0	1.1301E+33	0.0085	0.8013	0.0151	0.8115	0.8115
4.5	3.0535E+34	0.0076	0.7874	0.0152	0.8401	0.8401

**Minimaxloss due to one missing observation.

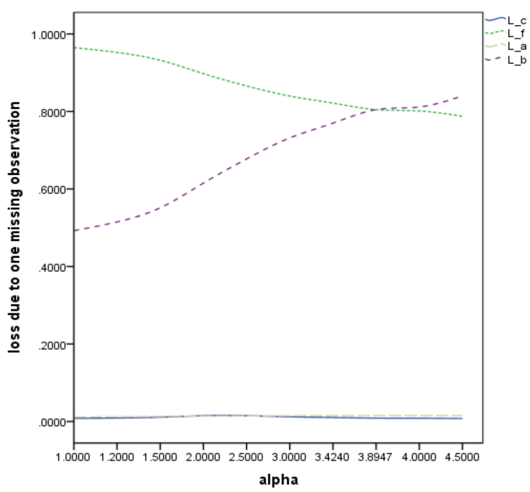

Figure 7: Loss curve due to one missing observation for a 5-factor D(1,4) split-plot CCD with n_f = 16 points from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 16, n_β = 8, and n_c = 8 points, plotted against α.

8. Five-Factor D(2,3) Split-plot CCD.

A five-factor split-plot CCD with two whole-plot and three subplot variables, consists of n_f = 16 points from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 16 points of whole-plot axial part, n_β = 6 points of subplot axial part and n_c = 4 center points. There are 10 whole plots and N = 42 total design points. The losses due to missing single observations of these different points are studied for range of α from 1.0 to 5.0, and are given in the table below.

Table 8: Losses due to Single Missing Observations of Different Design Points for D(2,3) Split-plot CCD.

A	$\lvert X'V^{-1}X \rvert$ for complete design	Losses due to missing a Center point (L_c)	Losses due to missing a factorial point (L_f)	Losses due to missing a whole- plot axial point (L_α)	Losses due to missing a subplot axial point (L_β)	Overall maxloss due to single missing observation
1.0	7.0957E+18	0.0235	0.9633	0.0410	0.4723	0.9633
1.20	1.7531E+20	0.0274	0.9508	0.0436	0.4953	0.9508
1.50	9.8195E+21	0.0365	0.9309	0.0469	0.5319	0.9309
2.0	2.4753E+24	0.0581	0.8967	0.0509	0.5949	0.8967
2.236	2.7729E+25	0.0624	0.8812	0.0525	0.6261	0.8812
2.5	4.0076E+26	0.0577	0.8661	0.0541	0.6612	0.8661
3.0	4.9255E+28	0.0405	0.8434	0.0564	0.7203	0.8434
4.0711	2.6058E+32	0.0230	0.8089**	0.0590	0.8089**	0.8089**
4.45	4.5837E+33	0.0203	0.7989	0.0596	0.8341	0.8341
5.0	9.5340E+35	0.0183	0.7891	0.0601	0.8581	0.8581

**Minimaxloss due to one missing observation.

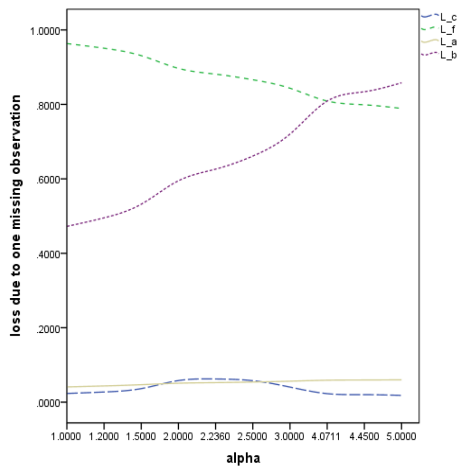

Figure 8: Loss curve due to one missing observation for a 5-factor D(2,3) split-plot CCD with n_f = 16 points from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 16, n_β = 6, and n_c = 4 points, plotted against α.

9. Six-Factor D(2,4) Split-plot CCD.

A six-factor split-plot CCD with two whole-plot and four subplot variables, consists of n_f = 32 points from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 32 points of whole-plot axial part, n_β = 8 points of subplot axial part and n_c = 8 center points. There are 10 whole plots and N = 80 total design points. The losses due to missing single observations of these different points are studied for range of α from 1.0 to 3.5, and are given in the table below.

Table 9: Losses due to Single Missing Observations of Different Design Points for D(2,4) Split-plot CCD.

A	$\lvert X'V^{-1}X \rvert$ for complete design	Losses due to missing a Center point (L_c)	Losses due to missing a factorial point (L_f)	Losses due to missing a whole- plot axial point (L_α)	Losses due to missing a subplot axial point (L_β)	Overall maxloss due to single missing observation
1.0	1.3467E+26	0.0063	0.6375	0.0103	0.4646	0.6375
1.20	6.8184E+33	0.0071	0.6335	0.0109	0.4774	0.6335
1.50	9.1339E+35	0.0088	0.6268	0.0118	0.4991	0.6268
2.0	6.4030E+37	0.0131	0.6144	0.0128	0.5400	0.6144
2.449	9.9419E+40	0.0156	0.6028	0.0135	0.5812	0.6028
2.5	1.7259E+41	0.0155	0.6015	0.0135	0.5861	0.6015
2.626	6.6580E+41	0.0152	0.5984**	0.0137	0.5984**	0.5984**
3.0	3.2322E+43	0.0125	0.5897	0.0141	0.6345	0.6345
3.5	3.9053E+45	0.0091	0.5791	0.0144	0.6787	0.6787

**Minimaxloss due to one missing observation.

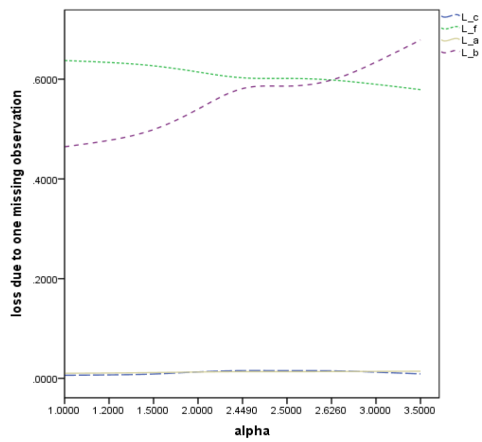

Figure 9: Loss curve due to one missing observation for a 6-factor D(2,4) split-plot CCD with n_f = 32 points from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 32, n_β = 8, and n_c = 8 points, plotted against α.

CONCLUSIONS

From the various types of split-plot central composite designs we have constructed based on their losses when a single observation is missing, the designs robust to a single missing observation are given in Tables 10 and 11 below for different configurations. Table 10 consists of the designs with single replication of factorial and axial parts. It can be seen from this table that the 5-factor split-plot CCD with 3 whole plot and 1 subplot variables and **α = β = 2.2007** has the minimum value of the maximum loss when an observation is missing, while a 3-factor split plot CCD with 1 whole plot and 2 subplot variables and **α = β = 1.9120** has the maximum loss for a single missing observation.

Table 11 consists of those with half replication of factorial part and complete replication of axial part. The losses due to one missing observation are given in the last column of each of the tables and these losses are the minimaxlosses for that configuration. We can see from this table that the 6-factor split-plot CCD with 2 whole plot and 4 subplot variables and **α = β = 2.626** has the minimum value of the maximum losses when an observation is missing, while a 5-factor split plot CCD with 1 whole plot and 4 subplot variables and **α = β = 3.424** has the maximum loss for missing a single observation.

Table 10: Split-plot Central Composite Designs with Single Replication of Factorial and Axial Parts Robust to a Single Missing Observation.

wp	k	w	s	n_f	n_α	n_β	n_c	N	α (= β)	$\lvert X'V^{-1}X \rvert$	Minimaxloss1
6	2	1	1	4	4	2	2	12	1.5946	0.66133E+4	0.6200
10	3	2	1	8	8	2	2	20	2.0320	0.12829E+9	0.5977
6	3	1	2	8	8	4	4	24	1.9120	0.37980E+10	0.6520
6	4	1	3	16	16	6	8	46	1.8560	0.15423E+18	0.5761
16	4	3	1	16	12	2	2	32	2.2007	0.17478E+17	0.4921
10	4	2	2	16	16	4	2	38	1.8750	0.11965E+17	0.5359

Table 11: Split-Plot Central Composite Designs with Half Replication of Factorial Part and Complete Replication of Axial Part Robust to a Single Missing Observation.

wp	K	W	s	n_f	n_α	n_β	n_c	N	α (= β)	$\lvert X'V^{-1}X \rvert$	Minimaxloss1
6	5	1	4	16	16	8	8	48	3.424	0.46033E+31	0.8221
10	5	2	3	16	16	6	4	42	4.0711	0.26058E+33	0.8089
10	6	2	4	32	32	8	8	80	2.626	0.66580E+42	0.5984

Our computations have shown that the loss due to missing an observation in these designs is robust to changes in the ratio of the two variance components, and also that, for a given k-factor split-plot central composite design, the loss corresponding to a missing point depends on the design's configuration, that is, on the number of whole plot and subplot factors. Therefore, if there is risk of a single missing observation in the experiment, then we hereby recommend for the practitioner, the minimaxloss1 designs developed for each configuration.

REFERENCES

1. Akhtar, M. and P. Prescott. 1986. "Response Surface Designs Robust to Missing Observations". *Commun. Statist. Simula*. 15(2):345-363.

2. Akhtar, M. 2001. "Five –Factor Central Composite design Robust to a Pair of Missing Observations". *Journal of Research (Science)*. Bahauddin Zakariya University, Multan, Pakistan.12(2):105-115.

3. Akram, M. 2002. "Central Composite Designs Robust to Three Missing Observations". Unpublished Ph.D. Thesis. The Islamia University: Bahawalpur, Pakistan.

4. Andrews, D.F. and A.M. Herzberg. 1979. "The Robustness and Optimality of Response Surface Designs". *J. Statist. Plan. Inf*. 3:249-257.

5. Anbari, F. and J. Lucas. 1994. "Supper–Efficient Designs: How to Run your Experiment for Higher Efficiency and Lower Cost". *ASQC Technical Conference Transactions*. 852-863.

6. Addelman, S. 1964. "Some Two-Level Factorial Plans with Confounding". *Technometrics*. 6:253 – 258.

7. Ahmad, T. and S.G. Gilmour. 2010. "Robustness of Subset Response Surface Designs to Missing Observations". *Journal of Statistical Planning and Inference*. 140:92 – 103.

8. Box, G.E.P. and K.B. Wilson. 1951. "On the Experimental Attainment of Optimum Conditions (with discussions)". *Journal of the Royal Statistical Society Series B*. 13(1):1-45.

9. Box, G.E.P., G. Hunter, J. Stuart, and G. William. 2005. *Statistics for Experimenters*. John Willey and Sons, Inc.: Princeton, NJ.

10. Box, G.E.P. and N.R. Drapper. 1975. "Robust Designs". *Biometrika*. 62(2):347 – 352.

11. Box, G.E.P. and N.R. Draper. 1963. "The Choice of a Second Order Rotatable Design". *Biometrika*. 50:335–352.

12. Box, G.E.P. and D.W. Behnken. 1960. "Some New Three Level Designs for the Study of Quantitative Variables". *Technometrics.* 2:455-475.

13. Box, G.E.P. and J.S. Hunter. 1957. "Multi-Factor Experimental Designs for Exploring Response Surfaces". *The Annals of Mathematical Statistics*. 28:195-241.

14. Daniel, C. 1976. *Applications of Statistics to Industrial Experiments*". Wiley: New York, NY.

15. Goos, P. and M. Vandebroek,. 2001. "Optimal Split-plot Designs". *Journal of Quality Technology*. 33:436-450.

16. Goos, P. and M. Vandebroek. 2003. "D-Optimal Split-plot Designs with Given Numbers and Sizes of Whole Plots". *Technometrics*. 45:235-245.

17. Goos, P. and M. Vandebroek. 2004. "Outperforming Completely Randomized Designs". *Journal of Quality Technology*. 36:12-26.

18. Ganju, J. and J.M. Lucas. 2005. "Randomized and Random-Run-Order Experiments". *Journal of Statistical Planning and Inference*. 133:199-210.

19. Gilmour, S.G. 2006. "Response Surface Designs for Experiments in Bioprocessing". *Biometrics*. 62:323-331.

20. Herzberg, A.M. and D.F. Andrews. 1976. "Some Considerations in the Optimal Design of Experiments in Non-Optimal Situations". *J.R.S.S. Series B*. 38:284-289.

21. Ju, L.H. and J.M. Lucas. 2002. "LK Factorial Experiments with Hard-To-Change and Easy-To-Change Factors". *Journal of Quality Technology*. 34(4):411-421.

22. Letsinger, J.D., R.H. Myers, and M. Lentner. 1996. "Response Surface Methods for Bi-Randomization Structures". *Journal of Quality Technology*. 28: 381-397.

23. Myers, R.H., D.C. Montgomery, M.A. Christine. 2009. *Response Surface Methodology, Process and Product Optimization Using Designed Experiments*. Wiley: New York, NY.

24. Kowalski, S.M., G.G. Vining, D.C. Montgomery, and C.M. Borror. 2006. "Modifying Central Composite Designs to Model the Process Mean and Variance when there are Hard-To-Change Factors. *J.R.S.S. Series C.*, 55(5):615-630.

25. Trinca, L.A. and S.G. Gilmour. 2001. "Multistratum Response Surface Designs". *Technometrics*. 43:25–33.

26. Vining, G.G., S.M. Kowalski, and D.C. Montgomery. 2005. "Response Surface Designs within a Split-Plot Structure". *Journal of Quality Technology*. 37:115-129.

27. Webb, D.F., J.M. Lucas, and J.J. Borkowski. 2004. "Factorial Experiments when Factor Levels are not Necessarily Reset". *Journal of Quality Technology*. 36(1):1-11.

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