The differential formulation of the Tau method and its error estimate for fourth order non-overdetermined differential equations

By

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Abstract: The Tau method has for some time been plagued with the problem of providing a computationally efficient general error estimation procedure for the perturbed problem. In this paper we are concerned with Differential formulation of the Tau methods for numerical solution of initial value problems in non-over determined fourth order ordinary differential equations. To this end, a polynomial is constructed based on the error function associated with polynomial economization which gives a theoretical estimate of the error of the Tau method. In doing so, the number of undetermined constants is kept to a minimum and the resulting polynomial does not require further evaluation in the interval under consideration. The error estimation formula obtained for the class of ODEs is efficient and accurate. Tau Numerical results and details of the algorithm confirm the high accuracy and user-friendly structure of this numerical approach.

Keywords: Approximation, Chebyshev, Perturbation, Polynomial, Variant

Introduction

The tau method first introduced by Lanczos (1938) has over time been developed into different variants so as to improve its accuracy, widen its scope of application or render it amenable for easier use. In this direction, Lanczos (1956) developed a modification based on the use of canonical polynomials and Ortiz (1974) showed that the elements of the canonical polynomial sequence can be generated by means of a simple recursion relation which is self-starting and explicit. With the aid of a certain procedure which Lanczos (1956) called "Tau method",approximation of high accuracy could be obtained for a number of functions used in scientific and engineering computations. Accurate approximate polynomial solutions of linear ordinary coefficients can be obtained by using the Tau method introduced by Lanczos (1956). Techniques based on this method have been reported in the in the literature with applications to more general equations including non-linear ones Onumanyi (1982), while techniques based on direct Chebyshev series replacement have been discussed by Fox (1981). We review here briefly the differential form of tau method.

Differential form of the Tau method

Consider the following boundary value problem in the class of m-th order ordinary differential equations.

$$Ly(x) :\equiv \sum_{r=0}^{m} P_r(x)\, y^{(r)}(x) = f(x), \quad a \le x \le b \qquad (1.1a)$$

$$L^{*}y(x_{rk}) \equiv \sum_{r=0}^{m} a_{rk}\, y^{(r)}(x_{rk}) = \rho_k, \quad k = 1(1)m \qquad (1.1b)$$

where $|a| < \infty$, $|b| < \infty$, a_{rk}, x_{rk}, ρ_k, $r = 0(1)m$, $k = 0(1)m$, are given real numbers, and the functions $f(x)$ and

$$P_r(x) = \sum_{k=0}^{N_r} P_{r,k}\, x^k, r = 0(1)m \qquad (1.2)$$

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are polynomial functions or sufficiently close polynomial approximation of given functions.

Definition

Equation (1.1a) is said to be non-overdetermined if s, given by (1.3) is zero, i.e. if s = 0, otherwise it is overdetermined. i.e. The number of over-determination, s, of equation (1.1a) is defined as :

$$s = \max\{N_r - r : 0 \le r \le m\}, \quad \text{for } N_r \ge r \text{ and } 0 \le r \le m \qquad (1.3)$$

For the solution of (1.1) by the tau method [seeAdeniyi,R.B(1991),Adeniyi and Onumanyi (1991),Lanczos(1956) and Ortiz(1974)]we seek an approximate.

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \quad n + \infty \qquad (1.4)$$

of $y(x)$ which satisfies exactly the perturbed problem

$$Ly_n(x) = f(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+1}(x), \quad a \le x \le b \qquad (1.5a)$$

$$L^* y_n(x_{rk}) = \rho_k, \quad k = 1(1)m \quad (1.5b)$$

Where, $r = 1(1)m + s$ are fixed parameters to be determined along with a_r, $r = 0(1)m$

In (1.4), by equating the coefficient of the power of x in (1.5).The polynomial

$$T_r(x) = \cos\left\{r \cos^{-1}\left[\frac{2x-2a}{b-a}\right]\right\} = \sum_{k=0}^{r} C_k^{(r)} x^k \qquad (1.6)$$

is the r-th degree Chebyshev polynomial valid in [a, b] (seeAdeniyi and Onumanyi(1991),Fox and Parker(1981) Ortiz(1974)

Error estimation of the Tau method

We shall discuss briefly the error estimation of the tau method for the differential variant.

Error estimation for the differential form

If we define the error function

$$e_n(x) = y(x) - y_n(x) \qquad (2.1)$$

which satisfies the error problem

$$Le_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+1}(x) \quad (2.2a)$$

$$Le_n(x_{rk}) = 0, k = 1(1)m (2.2b)$$

The polynomial error approximation

$$(e_n(x))_{n+1} = \frac{\mu_n \varphi_n T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \qquad (2.3)$$

of $e_n(x)$ satisfies the perturbed error problem

$$L(e_n(x))_{n+1} = \sum_{r=0}^{m+s-1} (-\tau_{m+s-1} T_{n-m+r+1}(x) + \hat{\tau}_{m+s-1} T_{n-m+r+2}(x)) \qquad (2.4a)$$

$$L^*(e_n(x_{rk}))_{n+1} = 0 \qquad (2.4b)$$

where the extra parameter $\hat{\tau}_r$, $r = 1(1)m + s$, and φ_n in (2.3) – (2.4) are to be determined and $\mu_m(x)$ in (2.3) is a specified polynomial of degree in which ensures that $(e_n(x))_{n+1}$ satisfies the homogenous conditions (2.4b). With (2.3) in (2.4), we get a linear system of $m + s + 1$ equations, obtained by equating the coefficients of $x^{n+s+1}, x^{n+s}, \dots, x^{n-m+1}$, for the determination of φ_n by forward elimination, since we do not need the $\hat{\tau}$'s in (2.3), consequently, we obtain an estimate.

$$\varepsilon = \max_{a \le x \le b} |(e_n(x))_{n+1}| = \frac{|\varphi_m|}{|C_{n-m+1}^{(n-m+1)}|} \cong \max_{a \le x \le b} |e_n(x)|$$

A class ofnon-overdetermined fourth order differential equations

We consider here the differential variant of the Tau method for the Tau approximants and their error estimates for the class of problems:

$$(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4) y^{iv}(x) + (\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3) y^{iii}(x) + (\gamma_0 + \gamma_1 x + \gamma_2 x^2) y^{ii}(x) + (\delta_0 + \delta_1 x) y'(x) + \omega_0 y(x) = \sum_{r=0}^{n} f_r x^r \ (3.1a)$$

$$y(a) = \rho_0, \ y'(a) = \rho_1, \ y''(a) = \rho_2, \ y'''(a) = \rho_3 (3.1b)$$

That is, in this case m = 4 and s = 0 in (1.1). We shall assume that a = 0 and b = 1 since

$$u = \frac{(x-a)}{b-a}, \quad a \le x \le b \qquad (3.2)$$

Transforms (3.1) into closed interval [0, 1].

Tau approximant by the differential form

In order to deriver tau approximant, we seek an approximate solution (1.4) and substitute into slightly perturbed (3.1) to have.

$$(\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3+\alpha_4 x^4)\sum_{r=0}^{n} r(r-1)(r-2)(r-3)a_r x^{r-4}+$$

$$(\beta_0+\beta_1 x+\beta_2 x^2+\beta_3 x^3)\textstyle\sum_{r=0}^{n} r(r-1)(r-2)a_r x^{r-3}+$$

$$(\gamma_0+\gamma_1 x+\gamma_2 x^2)\sum_{r=0}^{n} r(r-1)a_r x^{r-2}$$

$$+(\delta_0+\delta_1 x)\sum_{r=0}^{n} r\,a_r x^{r-2}+\omega_0\sum_{r=0}^{n} a_r x^r$$

$$=\ H_n(x)+\sum_{r=0}^{n} f_r x^r \qquad (3.3)$$

which yields ,

$$\sum_{r=0}^{n} r(r-1)(r-3)\alpha_0 a_r x^{r-4}$$

$$+\sum_{r=0}^{n}[r(r-1)(r-2)(r-3)\alpha_1 + r(r-1)(r-r(r-$$

$$r(r-1)(r-r(r-2)\beta_0]a_r x^{r-3}$$

$$+\sum_{r=0}^{n}[r(r-1)(r-2)(r-3)\,a_r x^{r-2}$$

$$+\sum_{r=0}^{n} r(r-1)(r-2)\beta_2+r(r-1)\gamma_1$$

$$+r\delta_0]\,a_r x^{r-1}+\sum_{r=0}^{n}[r(r-1)(r-2)(r-3)\alpha_4$$

$$+r(r-1)(r-2)\beta_3+r(r-1)\gamma_2+r\delta_1]a_r x^r$$

$$=\sum_{r=0}^{n} f_r x^r+\tau_1 T_n(x)+\tau_2 T_{n-1}(x)+\tau_3 T_{n-2}(x)$$

$$+\tau_4 T_{n-3}(x) \qquad (3.4)$$

where,

$$T_n(x)=\sum_{r=0}^{n} C_r^{(n)} x^r,\ T_{n-1}(x)$$

$$=\sum_{r=0}^{n-1} C_r^{(n-1)} x^r,\ T_{n-2}(x)=\sum_{r=0}^{n-2} C_r^{(n-2)} x^r,$$

$$T_{n-3}(x)=\sum_{r=0}^{n-3} C_r^{(n-3)} x^r,$$

This leads to:

$$\sum_{k=0}^{n}[\alpha_4 k(k-1)(k-2)(k-3)(k-2)$$

$$+\gamma_2 k(k-1)+\delta_1 k+\omega_0]a_k x^k$$

$$+\sum_{k=0}^{n-1}[\alpha_3+\beta_2 k(k+1)(k-1)+\gamma_1 k(k+1)$$

$$+\delta_0(k+\beta_2 k(k+1)(k-1)+\gamma_1 k(k+1)$$

$$+\delta_0(k+1)]\,a_{k+1}x^k$$

$$+\textstyle\sum_{k=0}^{n-2}[\,\alpha_2(k+2)(k+1)k(k-2)+$$

$$\beta_1(k+2)(k+1)k+\gamma_0(k+2)(k+1)]a_{k+2}x^k$$

$$+\sum_{k=3}^{n-3}[\alpha_1(k+3)(k+2)(k+1)k+$$

$$+\sum_{k=0}^{n-4}[\alpha_0(k+4)[(k+3)(k+2)(k+1)]$$

$$a_{k+4}x^k+\ \tau_1\textstyle\sum_{k=0}^{n} C_k^{(n)} x^k+\ \tau_2\sum_{k=0}^{n-1} C_k^{(n-1)} x^k+$$

$$+\ \tau_3\textstyle\sum_{k=0}^{n-2} C_k^{(n-2)} x^k+\ \tau_4\sum_{k=0}^{n-3} C_k^{(n-3)} x^k \quad (3.5)$$

hence,

$$[\alpha_4 n(n-1)(n-2)(n-3)+\beta_3 n(n-1)-$$

$$(n-2)+\gamma_2(n-1)+\delta_1 n+\omega_0]a_n-f_n$$

$$\tau_1 C_n^{(n)}\}x^n+\{[\alpha_4 n(n-1)(n-2)(n-3)(n-4)$$

$$+\beta_3 n(n-1)(n-2)(n-3)\ +\gamma_2(n-1)(n-2)$$

$$+\ \delta_1(n-1)+\omega_0]\ a_{n-1}$$

$$+[\alpha_3 n(n-1)(n-2)(n-3)+\beta_2 n(n-1)$$

$$-\tau_1 C_{n-1}^{(n)}-\tau_2 C_{n-1}^{(n-1)}\}x^{n-1}\{[\alpha_4(n-2)(n-3)$$

$$(n-2)+\gamma_1 n(n-1)+[\delta_0 n]a_n f_{n-1}[(n-4)$$

$$(n-5)+\beta_3(n-2)(n-3)(n-4)\gamma_2$$

$(n-2)(n-3)+\delta_1(n-2)+\omega_0]\}\, a_{n-2}$
$+[\alpha_3(n-1)(n-2)(n-3)(n-4)$
$+\beta_2(n-1)(n-2)(n-3)+\gamma_1(n-1)$
$(n-2)+\delta_0(n-1)]a_{n-1}+[\alpha_2 n(n-1)$
$(n-2)(n-3)+\beta_1 n(n-1)(n-2)+\gamma_0$
$n(n-1)]a_n f_{n-2}-\tau_1 C_{n-2}^{(n)}-\tau_2 C_{n-2}^{(n-1)}-$
$\tau_3 C_{n-2}^{(n-2)}\}x^{n-2}+\{[\alpha_4(n-3)(n-4)(n-5)(n-$
$6)+\beta_3(n-3)(n-4)(n-5)$
$+\gamma_2(n-3)(n-4)+\delta_1(n-3)+\omega_0]a_{n-3}$
$+[\alpha_3(n-2)(n-3)(n-4)(n-5)+\beta_2$
$(n-2)(n-3)(n-4)+\gamma_1(n-2)(n-3)$
$+\delta_0(n-2)]\, a_{n-2}[\alpha_2(n-1)(n-2)(n-3)$
$(n-4)+\beta_1(n-1)(n-2)(n-3)+\gamma_0$
$(n-1)(n-2)]\, a_{n-1}+[\alpha_1 n(n-1)(n-2)$
$(n-3)+\beta_0 n(n-1)(n-2)]\, a_n f_{n-3}-$
$-\tau_3 C_{n-3}^{(n-2)}-\tau_4 C_{n-3}^{(n-3)}-f_{n-3}=0 \quad (3.7b)$

$[\alpha_4(n-2)(n-3)(n-4)(n-5)+\beta_3$
$(n-2)(n-3)(n-4)+\gamma_2(n-2)(n-3$
$)+\delta_1(n-2)+\omega_0]a_{n-2}+[\alpha_3(n-1)$
$(n-2)(n-3)(n-4)+\beta_2(n-1)$
$(n-2)(n-3)+\gamma_1(n-1)(n-2)+\delta_0$
$(n-1)]a_{n-1}+[\alpha_2 n(n-1)(n-2)(n-3)$
$+\beta_1 n(n-1)(n-2)+\gamma_0 n(n-1)]a_n-$
$\tau_1 C_{n-2}^{(n)}-\tau_2 C_{n-2}^{(n-1)}-\tau_3 C_{n-2}^{(n-2)}-f_{n-2}=0$

(3.7c)
$[\alpha_4(n-1)(n-2)(n-3)(n-4)+\beta_3$
$(n-1)(n-2)(n-3)+\gamma_2(n-1)(n-2)$
$+\delta_1(n-1)+\omega_0]a_{n-1}+[\alpha_3 n(n-1)$
$(n-2)(n-3)+\beta_2 n(n-1)(n-2)+\gamma_1$
$n(n-1)+\delta_0 n]a_n-\tau_1 C_{n-1}^{(n)}-\tau_2 C_{n-1}^{(n-1)}$
$-f_{n-1}=0 \quad (3.7d)$

$[\alpha_4 n(n-1)(n-2)(n-3)+\beta_3 n(n-1)$
$(n-2)+\gamma_2 n(n-1)+\delta_1 n+\omega_0]a_n$
$-\tau_1 C_n^{(n)}-f_{n-1}=0 \quad (3.7e)$

The solution of the system together with the three equations arising from the condition (3.1b) for $a_r,\ r=0(1)n$ and $\tau_1,\tau_2,\tau_3,\tau_4$, subsequently leads to the approximant $y_n(x)$

Error estimation for differential form

For problem (3.1) we have from 2.4

$\tau_1 C_{n-3}^{(n)}-\tau_2 C_{n-3}^{(n-1)}-\tau_3 C_{n-3}^{(n-2)}-\tau_4 C_{n-3}^{(n-3)}\}$
$x^{n-3}+\alpha_0 n(n+1)(n-1)(n-2)a_{n+1}=0$

(3.6)

Here, we obtain by equating corresponding coefficients of the linear system to have

$\{[\alpha_4 k(k-1)(k-2)(k-3)+k(k-1)$
$(k-2)+\gamma_2 k(n-1)+\delta_1 k+\omega_0]a_k+[\alpha_3$
$(k+1)k(k-1)(k-2)+\beta_2 k(k+1)k$
$(k-1)+\gamma_1(k+1)k+\delta_0(k+1)]a_{k+1}$
$+[\alpha_2(k+2)(k+1)k(k-2)+\beta_1(k+2)$
$(k+1)k+\gamma_0(k+2)(k+1)]a_{k+2}$
$+[\alpha_1(k+3)(k+2)(k+1)k+\beta_0(k+3)$
$(k+2)(k+1)]\, a_{k+3}-\tau_1 C_k^{(n)}-\tau_2 C_k^{(n-1)}$
$L(e_n(x))_{n+1}\equiv \hat{\tau}_1 T_{n+1}(x)+(\hat{\tau}_3-\hat{\tau}_2)T_{n-1}(x)$
$+(\hat{\tau}_4-\hat{\tau}_3)T_{n-2}(x)-\tau_4 T_{n-3}(x) \quad (3.8a)$

where ,

$$(e_n(x))_{n+1}=\frac{x^4\phi_n T_{n-3}(x)}{C_{n-3}^{(n-3)}}$$
$$=\frac{\phi_n\sum_{r=0}^{n-3}C_r^{(n-3)}x^{r+4}}{C_{n-3}^{(n-3)}} \quad (3.8b)$$

differentiating (3.4b) and substitute into (3.4a) to have

$$(\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3+\alpha_4 x^4)\frac{\phi_n}{C_{n-3}^{(n-3)}}\sum_{r=0}^{n-3}(r+1)(r+2)(r+3)(r+4)C_r^{(n-3)}x^r$$
$$+(\beta_0+\beta_1 x+\beta_2 x^2+\beta_3 x^3)\frac{\phi_n}{C_{n-3}^{(n-3)}}\sum_{r=0}^{n-3}(r+2)(r+3)(r+4)C_r^{(n-3)}x^{r+1}+(\gamma_0$$
$$+\gamma_1 x+\gamma_2 x^2)\frac{\phi_n}{C_{n-3}^{(n-3)}}\sum_{r=0}^{n-3}(r+3)(r+4)$$
$$C_r^{(n-3)}x^{r+2}+(\delta_0+\delta_1 x)\sum_{r=0}^{n-3}(r+4)\frac{\phi_n}{C_{n-3}^{(n-3)}}$$
$$C_r^{(n-3)}x^{r+3}+\omega_0\frac{\phi_n}{C_{n-3}^{(n-3)}}\sum_{r=0}^{n-3}C_r^{(n-3)}x^{r+3}$$

(3.8c)

Thus, from the coefficient of $x^{n+1}, x^{n}, x^{n-1}, x^{n-2}$ and and x^{n-3}, we get the system

$$\hat{\tau}_1 C_{n+1}^{(n+1)} = \frac{\phi_n}{C_{n-3}^{(n-3)}}[\omega_0 + (n+1)\delta_1 + (n+1)n\gamma_2 + (n-1)n(n+1)\beta_3 + (n-2)(n-1)n(n+1)\,\alpha_4]C_{n-3}^{(n-3)} \quad (3.9a)$$

$$\hat{\tau}_1 C_{n}^{(n+1)} + (\hat{\tau}_2 - \tau_1)C_n^{(n)} = \frac{\phi_n}{C_{n-3}^{(n-3)}}[\{(n+1)\delta_1 + (n+1)n\gamma_2 + (n-1)n(n+1)\beta_2 + (n+1)n(n-1)(n-2)\alpha_3\}C_{n-3}^{(n-3)}\ \{\omega_0 + n(n-1)\gamma_2 + n\delta_1 + (n+1)n(n-1)\beta_3\}C_{n-4}^{(n-3)}] \quad (3.9b)$$

$$\hat{\tau}_1 C_{n}^{(n+1)} + (\hat{\tau}_2 - \tau_1)C_{n-1}^{(n)} = \frac{\phi_n}{C_{n-3}^{(n-3)}}[\{(n+1)n(n-1)\beta_1 + (n+1) + (\hat{\tau}_3 - \tau_2)C_{n-1}^{(n-1)}\{n(n-1)(n-2)\alpha_2\}C_{n-3}^{(n-3)} + \{n(n-1)(n-2)\beta_2 + n(n-1)\gamma_0 + n(n+1)\gamma_1 + n\delta_0\}C_{n-4}^{(n-3)} + \{(n-1)(n-2)\gamma_2 + (n-1)\delta_1 + \omega_0\}C_{n-5}^{(n-3)} \quad (3.9c)$$

$$\hat{\tau}_1 C_{n-2}^{(n+1)} + (\hat{\tau}_2 - \tau_1)C_{n-2}^{(n)} + (\hat{\tau}_3 - \tau_1)C_{n-2}^{(n-1)} + (\hat{\tau}_4 - \tau_3)C_{n-2}^{(n-2)} = \frac{\phi_n}{C_{n-3}^{(n-3)}}[\{n(n+1)\{(n-1)\beta_0 + (n+1)n(n-1)(n-2)\alpha_1\}C_{n-3}^{(n-3)} + \{n(n-1)\gamma_0 + n(n-1)(n-2)\beta_1\}C_{n-4}^{(n-3)} + (n-1)\delta_0\{(n-1)\delta_0 + (n-1)(n-2)\gamma_1\} + (n-1)(n-2)\gamma_1 C_{n-5}^{(n-3)}] \quad (3.9d)$$

$$\hat{\tau}_1 C_{n-3}^{(n+1)} + (\hat{\tau}_2 - \tau_1)C_{n-3}^{(n)} + (\hat{\tau}_3 - \tau_2)C_{n-3}^{(n-1)} + (\hat{\tau}_4 - \tau_3)C_{n-3}^{(n-2)} - \tau_4 C_{n-3}^{(n-3)} = \frac{\phi_n}{C_{n-3}^{(n-3)}}[\{(n+1)n(n-1)(n-2)\alpha_0 C_{n-3}^{(n-3)} + (n+1)n(n-1)\beta_0 C_{n-4}^{(n-3)} + (n-1)(n-2)\gamma_0 C_{n-5}^{(n-3)}] \quad (3.9e)$$

Using well-known relation,

$C_n^{(n)} = 2^{2n-1}$, $C_{n-1}^{(n)} = -\frac{1}{2}nC_n^{(n)}$, $C_{n-1}^{(n)} = -n * 2^{2n-2}$. We solve this system by forward substitution to for φ_n to obtain.

$$\phi_n = \frac{2^{2n-7}\tau_4}{k_6} \quad (3.10)$$

where,

$$k_6 = \frac{k_2}{C_n^{(n)}} - \frac{C_n^{(n+1)}C_{n-3}^{(n)}}{C_n^{(n)}C_{n+1}^{(n+1)}}k_1 - \frac{C_{n-3}^{(n-1)}k_3}{C_{n-1}^{(n-1)}} + \frac{C_{n-1}^{(n+1)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}}k_1 + \frac{C_{n-1}^{(n)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_n^{(n)}}k_2 + \frac{C_{n-1}^{(n+1)}C_{n-1}^{(n)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}}k_1 - \frac{C_{n-3}^{(n-2)}}{C_{n-2}^{(n-2)}}$$

$$\left\{\frac{C_{n-3}^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{C_{n-3}^{(n+1)}C_{n-2}^{(n)}}{C_{n+1}^{(n+1)}} - \frac{C_{n+1}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n+1}^{(n-1)}C_{n-1}^{(n-1)}} + \frac{C_n^{(n+1)}C_n^{(n)}C_{n-2}^{(n-1)}}{C_{n+1}^{(n+1)}C_{n-1}^{(n-1)}}\right\}k_1 - \left\{\frac{C_{n-2}^{(n)}}{C_n^{(n)}} + \frac{C_{n-1}^{(n)}C_{n-2}^{(n)}}{C_n^{(n)}C_{n-1}^{(n-1)}}\right\}K_2 = \frac{C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}}k_3 + k_4 - k_5 \quad (3.10b)$$

where,

$$K_1 = \omega_0 + (n+1)\delta_1 + n(n+1)\gamma_2 + n(n+1)(n-1)\beta_3 + n(n+1)(n-1)(n-2)\alpha_4$$

$$K_2 = (n+1)\delta_1 + n(n+1)\gamma_1 + n(n+1)(n-1)\beta_2 + n(n+1)(n-1)(n-2)\alpha_3 + [\omega_0 + n(n-1)\gamma_2 + n\delta_1 + n(n-1)(n-2)\beta_3]\frac{C_{n-1}^{(n-3)}}{C_{n-3}^{(n-3)}}$$

$$K_3 = [n(n+1)(n-1)\beta_1 + n(n+1)(n-1)(n-2)\alpha_2] + [n(n-1)(n-2)\beta_2 + n(n-1)\gamma_0 + [+n(n-1)\gamma_1 + n\delta_0]\frac{C_{n-1}^{(n-3)}}{C_{n-3}^{(n-3)}} + [(n-1)(n-2)\gamma_2 + (n-1)\delta_1 + \omega_0]\frac{C_{n-5}^{(n-3)}}{C_{n-3}^{(n-3)}}$$

$$K_4 = n(n+1)(n-1)\beta_0 + n(n+1)(n-1)$$

$$(n-2)\alpha_1 + [n(n-1)\gamma_0 + n(n-1)(n-2)$$

$$\beta_1]\frac{C_{n-1}^{(n-3)}}{C_{n-3}^{(n-3)}} + [(n-1)\delta_0 + (n-1)(n-2)\gamma_1]\frac{C_{n-5}^{(n-3)}}{C_{n-3}^{(n-3)}}$$

$$K_5 = n(n+1)(n-1)(n-2)\alpha_0 + n(n+1)$$

$$(n-1)\beta_0\frac{C_{n-4}^{(n-3)}}{C_{n-3}^{(n-3)}} + (n-1)(n-2)\alpha_0\frac{C_{n-5}^{(n-3)}}{C_{n-3}^{(n-3)}} \qquad (3.11)$$

Thus, from (2.5) we have,

$$\varepsilon = \frac{|\tau_4|}{|2K_6|} \qquad (3.12)$$

as our desired error estimate.

Numerical examples

We considered here three selected problems for experimentation with our results of the preceding sections. The exact errors are obtained as

$\varepsilon * = \max_{0\le x\le 1}\{|y(x_k) - y_n(x_k)|\}, \quad 0 \le x \le 1, \{x_k\} = \{0.01k\} \quad for\ k = 0(1)100$

Problem 4.1

$$Ly(x) = y^{iv}(x) + y'''(x) - 7y''(x) - y'(x) + 6y(x) = 0$$

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,\ y'''(0) = -1$$

With exact solution

$$y = \frac{11}{8}e^{x} - \frac{5}{12}e^{-x} - \frac{1}{8}e^{-3x}, \quad 0 \le x \le 1$$

The numerical results are presented in Table 1 below.

Table 1: Error and error estimates for problem 1

Error	Degree (n) 4	5	6	7
ε	2.50×10^{-2}	6.81×10^{-4}	2.03×10^{-6}	2.94×10^{-8}
$\varepsilon *$	1.03×10^{-1}	4.63×10^{-3}	3.56×10^{-5}	2.89×10^{-7}

Comment: ε^*accurately captures ε

Problem 4.2

$$Ly(x) = y^{iv}(x) - y(x) = 0, \quad 0 \le x \le 1$$

$$y(0) = \frac{7}{2}, y'(0) = -4, y''(0) = \frac{5}{2}, y'''(0) = -2$$

with exact solution

$$y = 3e^{-x} + \frac{1}{2}\cos x - \sin x$$

The numerical results are presented in Table 2 below:

Table 2: Error and error estimates for problem 4.2

Error	Degree (n) 4	5	6	7
ε	2.77×10^{-3}	1.98×10^{-6}	2.61×10^{-7}	2.45×10^{-10}
$\varepsilon *$	4.67×10^{-2}	3.49×10^{-5}	9.32×10^{-5}	1.23×10^{-9}

Comment: error estimate yields a better accuracy

Problem 4.3

$$Ly(x) = y^{iv}(x) - 4y(x) = 0, 0 \le x \le 1$$

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = 0, y'''(0) = 2$$

With analytic solutions

$$y = \frac{e^{\sqrt{2x}} + e^{-\sqrt{2x}}}{2}, \quad 0 \le x \le 1$$

The numerical results are presented in Table 3below:

Table 3: Error and error estimates for problem 4.S3

	Degree (n)			
Error	4	5	6	7
ε	7.55×10^{-4}	2.9710^{-6}	3.58×10^{-7}	1.29×10^{-9}
ε*	2.76×10^{-3}	7.66×10^{-5}	1.72×10^{-7}	6.31×10^{-9}

Comment: Order of approximant is captured.
Numerical experiments have been carried out in order to test the effectiveness of the choice.

The result obtained in the work shows the closeness between the error of Lanczos economization process and the error of Tau method. The numerical experiments support the theoretical results.Some theoretical results are given that simplify the application of the Tau Method. The application of the Tau Method to the numerical solution of such problems is shown. Numerical results and details of the algorithm confirm the high accuracy and user-friendly structure of this numerical approach.

Conclusion

The differential form of the tau method for the solution of initial value problem (IVPS) for fourth order non-over determined ordinary differential equation has been presented.We note that the present error estimate yields a better accuracy than the estimate of Lanczos (1956), Fox (1962), Onumanyi (1982).The Lanczos (1956) error estimation procedure is applicable to the class of first order linear ordinary differential equations with polynomial coefficients and whose solutions are defined in the interval [0,1]. The procedure is restricted to first order differential system which is not good enough. The method of Fox can handle similar problems of order one and of higher orders other than one but is not general in scope of its application. The practical error estimation procedure of Onumanyi and Ortiz (1982) gives very accurate estimates. This is due to the idea of the Tau method.Though, the procedure is not economical considering the cost of computing because it involves the inversion of a matrix of dimension of at least (m + s) dimension. The present error estimation technique shows a remarkable improvement over these works done on the subject of error analysis of the Tau method as it leads to error estimation formula with wider scope of application. Also, the estimate proposed here does not involve any iteration for linear problems nor matrix inversion. The results obtained in the present work demonstrate the closeness between the exact error and the approximate of the tau method, thus error estimate of the Tau method is effective and reliable.

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