Integral variant of the Tau methods for ordinary differential equations (IVPs) involving maximum of four Tau parameters

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Abstract: This paper concerns the Lanczos' Tau Method for the numerical solution of Ordinary Differential Equations (ODE).The integral variant of the Tau method is considered here. The general expressions for elements of the Tau matrix equation involved in the integrated variant of the Tau method for the m-th order linear ODE and the corresponding general error estimates for the class were obtained. Perturbing the integrated error equation improved the accuracy of the estimate significantly. The error estimation was based on the error of the Lanczos economization process and it satisfies the Corresponding Perturbed Differential Equation(PEDE).We integrate through this PEDE and consequently increased the order of the perturbation term leading to an increased in the accuracy of the result obtained. Members of the class of problems characterized by m+s= 4, were investigated for study,where m and s are the order of the differential equations and the number of over determination, respectively. Consequently, a generalized tau matrix system was constructed for the m-th order linear ODEs and a generalized error estimates for the class of problems with maximum of three over determinations were obtained

Keywords: Approximant, error estimate, order, over-determination, perturbation, Tau matrix

Introduction

In 1938, Lanczos introduced an accurate approximate solution for ordinary differential equations with polynomial coefficients using Tau method. The method is related to the principle of economization of a differentiable function implicitly defined by linear differential equations with polynomial coefficients.

To illustrate the tau method, let us consider the m-th order linear differential equation

$$Ly(x) := \sum_{i=0}^{m} P_i(x)y^{(i)}(x) = f(x) \qquad (1.1a).$$

with the smooth solution $y(x)$, $a \le x \le b$, $|a| < \infty$, $|b| < \infty$satisfying a set of multi-point boundary conditions

$$L^*y(x_{ij}) := \sum_{i=0}^{m-1} a_{ij}y^{(i)}(x_{ij}) = \alpha_j j = 1(1)m \qquad (1.1b)$$

where a_{ij} , x_{ij} , α_j i = 1(1)m-1 , j = 1(1)m are given real numbers , f (x) and $P_i(x)$, i= 0(1)m in (1.1) are polynomials.
The idea of Lanczos is to approximate the solution of the differential system(1.1) by n-th degree polynomial function

$$y_n(x) = \sum_{r=0}^{n} a_r x^r n < \infty \qquad (1.2)$$

which is the exact solution of a perturbed equation obtained by adding to the right hand side of (1.1a) a polynomial perturbation term. The polynomial $y_n(x)$, satisfies , the differential equation

$$Ly_n(x) := \sum_{i=0}^{m} P_i(x) y_n^{(i)}(x) = f(x) + H_n(x) \quad (1.3a)$$

$$L^{*} y_n(x_{ij}) := \sum_{i=0}^{m-1} a_{ij} y^{(i)}(x_{ij}) = \alpha_j j = 1(1)m \quad (1.3b)$$

where the perturbation term, $H_n(x)$ is constructed in such a way that (1.3) has a polynomial solution of degree n. From Lanczos (1938), $H_n(x)$ is taken as the linear combination of powers of x multiplied by chebyshev polynomials .This choice of the chebyshev polynomials arises from the desire to distribute the error defined by

$$\text{error} = \max_{a \le x \le b} |y(x) - y_n(x)| \quad (1.4)$$

evenly distributed in the interval [a,b].

The chebyshev polynomial $T_r(x)$ in $a \le x \le b$ is defined as

$$T_r(x) = \cos\left\{r \cos^{-1}\left[\left\{\frac{2(x-a)}{b-a}\right\} - 1\right]\right\} \equiv \sum_{k=0}^{r} C_k^{(r)} x^k \quad (1.5)$$

with,

$$C_k^{(r)} = 2^{2r-1}(b-a)^{-r} \quad (1.6)$$

From the point of view of accuracy , the form

$$H_n(x) := \sum_{i=0}^{m+s-1} \tau_{m+s-1} T_{n-m+i+2}(x) \quad (1.7a)$$

is considered throughout this work , where s denotes the number of overdetermination of (1.1a) defined by

$$S = \max\{N_r - r, 0 \le r \le m\} \text{ for } N_r \ge r \text{ and } 0 \le r \le m \quad (1.7b)$$

To determine the coefficients a_r , r = 0(1)n in $y_n(x)$ from (1.1.3)where is it , a system of linear algebraic equations $A\underline{\tau}$ = B, obtained by equating corresponding coefficients of like terms of powers of x in (1.3)not seen and then using conditions (1.3b), is solved , $\underline{\tau} = (a_0, a_1, a_2, \dots \dots a_n, \tau_1, \tau_2, \tau_3, \dots \dots, \tau_{m+s})^T$.

The tau method is of order p, in the sense that if the exact solution of (1.1) is itself a polynomial of degree less or equal to p, the method will reproduce it.(see Ortiz (1974). Techniques based on the method have been reported in the literature with application to more general equations including non-linear ones(see Onumanyi and Ortiz (1982) and Ortiz (1969))., while techniques based on direct chebyshev series replacement have been discussed by Fox (1968).

Definition of terms:

Definition 1.1

The differential system (1.3) will be called the Tau problem corresponding to the differential system (1.1).We shall call n –th degree polynomial $y_n(x)$, which satisfies the Tau problem (1.3), the tau approximant of (1.1) and the Tau solution of (1.3)

Definition 1.2

The system of equation $A\underline{\tau}$ = B, where $\underline{\tau} = (a_0, a_1, a_2, \dots \dots a_n, \tau_1, \tau_2, \tau_3, \dots \dots, \tau_{m+s})^T$, resulting from the process of solution of (1.3) will be referred to as the Tau system of (1.3).

Definition1.3

Overdetermination Fox (1968)

Consider the differential system

$$x^2 y'' - y(x) = 0, \quad 0 \le x \le 1, \quad (1.8a)$$

$$y(1) = 1, \quad (1.8b)$$

If we assume the polynomial solution (1.2) not seen for (1.1.8) , we can obtain the following equations.

$$-a_0 = 0$$
$$-a_0 = 0$$
$$ra_r - a_{r+1} = 0$$
$$na_n = 0 \quad (1.9) \quad r = 1(1)n-1$$

Since the last equation in (1.9) refers to the coefficient of x^{n+1}, we need to add a term$\tau_1 T_{n+1}^{*}(x)$to the right hand side of (1.8a) for the satisfaction of (1.9) .This is not, however, sufficient for the unique determination of all the coefficients as a_1 can be computed in two different ways from (1.9) and is thus over determined. So the number of over determination of (1.8a) is one. Actually, we have $(n+3)$ equations at our disposal - $(n+2)$ equations from equations from (1.9) and one equation from

(1.8b) - for the determination of (n+1) coefficients. Hence we need to introduce two other unknowns $\tau_1 and \tau_2$ in the perturbing terms at the right hand side of (1.8a). The right hand side of (1.8a) thus becomes $\tau_1 T^*_{n+1}(x) + \tau_1 T^*_n(x)$

The integrated formulation of Lanczos Tau method

Let us consider the m –th order linear differential system (1.1), that is ,

$$Ly(x) := \sum_{i=0}^{m} P_i(x) y^{(i)}(x) = f(x) \quad , a \le x \le b \qquad (1.10a)$$

$$L^* y(x_{ij}) := \sum_{i=0}^{m-1} a_{ij} y^{(i)}(x_{ij}) = \alpha_j \ , \ j = 1(1) \qquad (1.10b)$$

Let $\iiint \underline{\quad}^{i} \int g(x)dx$

denote the indefinite i times applied to the function g(x), and let

$$I_L = \iiint \underline{\quad}^{m} \int L(.)dx (1.12)$$

The integrated form of (1.10a) is then ,

$$I_L y(x) = \iiint \underline{\quad}^{m} \int f(x)dx + C_m(x) \qquad (1.13)$$

where $C_m(x)$ denotes an arbitrary polynomial of degree $(m-1)$ arising from constant of integration .The Tau approximant , $y_n(x)$, of (1.10) thus satisfies the perturbed problem

$$I_L y_n(x) = \iiint \underline{\quad}^{m} \int f(x)dx + C_m(x) + H_{m+n}(x) \qquad (1.14a)$$

$$L^* y_n(x_{ij}) = \alpha_j \ , \quad j = 1(1)m \qquad (1.14b)$$

where,

$$H_{n+m}(x) = \sum_{i=0}^{m+s-1} \tau_{m+s-1} T_{n+m+i+2}(x) \qquad (1.15)$$

The Tau problem (1.14) often gives a more accurate approximation than (1.3) due to higher perturbation term $H_{n+m}(x)$ in (1.14) [see Fox(1968) and Ortiz(1969)]

Derivation of Tau approximant

We consider here, the derivation of tau approximant for the integrated formulation of the tau method for the class of problem (1.1) where m+s = 4 (i.e for the cases where m = 1, s =3, m = 2, s = 2, m = 3, s = 1 and m= 4 , s = 0)

The Case m =1 , s = 3

For this case we have from (1.1) that,

$$Ly(x) = (P_{10} + P_{11}x + P_{12}x^2 + P_{13}x^3 + P_{14}x^4)y(x)' +$$

$$(P_{00} + P_{01}x + P_{02}x^2 + P_{03}x^3)\, y(x) \equiv \sum_{i=0}^{F} f_i x^i \ , \ y(a) = \alpha_0 \qquad (2.1)$$

and from (1.13) we have,

$$\int_0^x (P_{10} + P_{11}t + P_{12}t^2 + P_{13}t^3 + P_{14}t^4)\, y(t)' \,dt + \int_0^x (P_{00} + P_{01}t + P_{02}t^2 + P_{03}t^3) y(t)dt$$

$$= \int_0^x \left(\sum_{i=0}^{n} f_i t^i\right)dt + H_n(x) \qquad (2.2)$$

where,

$$H_n(x) = \tau_1 \sum_{r=0}^{n+4} C_r^{(n+4)} x^r + \tau_2 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r + \tau_3 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r + \tau_4 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r \qquad (2.3)$$

We seek an approximate solution of the form,

$$y_n(x) = \sum_{r=0}^{n} a_r x^r \qquad (2.4)$$

which gives,

$$\int_0^x (P_{10} + P_{11}t + P_{12}t^2 + P_{13}t^3 + P_{14}t^4) y_n(t)'\,dt + \int_0^x (P_{00} + P_{01}t + P_{02}t^2 + P_{03}t^3) y_n(t)\,dt$$

$$= \int_0^x \left(\sum_{i=0}^{n} f_i t^i\right) dt + \tau_1 \sum_{r=0}^{n+4} C_r^{(n+4)} x^r + \tau_2 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r + \tau_3 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r +$$

$$\tau_4 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r \qquad (2.5)$$

Integrating the terms of (2.5) , applying (1.1b) and collecting the like terms we obtain ,

$$P_{10} \sum_{r=0}^{n} a_r x^r + \sum_{r=0}^{n} \left[\frac{P_{00} + rP_{11}}{r+1}\right] a_r x^{r+1} + \sum_{r=0}^{n} \left[\frac{P_{01} + rP_{12}}{r+2}\right] a_r x^{r+2} + \sum_{r=0}^{n} \left[\frac{P_{02} + rP_{13}}{r+3}\right] a_r x^{r+3}$$

$$+ \sum_{r=0}^{n} \left[\frac{P_{03} + rP_{14}}{r+4}\right] a_r x^{r+4} - \tau_1 \sum_{r=0}^{n+4} C_r^{(n+4)} x^r - \tau_2 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r - \tau_3 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r -$$

$$\tau_4 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r = P_{10} \propto_0 + \sum_{r=0}^{n} f_r \frac{x^{r+1}}{r+1} \qquad (2.6)$$

Equating the corresponding powers of x in (2.6) above, we obtain the tau system.

$A\tau = b$, where,

$$A = \begin{pmatrix}
P_{10} & 0 & 0 & 0 & 0 & 0 & -C_0^{(9)} & -C_0^{(8)} & -C_0^{(7)} & -C_0^{(6)} \\
P_{00} & P_{10} & 0 & 0 & 0 & 0 & -C_1^{(9)} & -C_1^{(8)} & -C_1^{(7)} & -C_1^{(6)} \\
\frac{P_{01}}{2} & \frac{P_{00}+P_{11}}{2} & P_{10} & 0 & 0 & 0 & -C_2^{(9)} & -C_2^{(8)} & -C_2^{(7)} & -C_2^{(6)} \\
\frac{P_{02}}{3} & \frac{P_{01}+P_{12}}{3} & \frac{P_{00}+2P_{11}}{3} & P_{10} & 0 & 0 & -C_3^{(9)} & -C_3^{(8)} & -C_3^{(7)} & -C_3^{(6)} \\
\frac{P_{03}}{4} & \frac{P_{02}+P_{13}}{4} & \frac{P_{00}+2P_{11}}{4} & \frac{P_{00}+3P_{11}}{4} & P_{10} & 0 & -C_4^{(9)} & -C_4^{(8)} & -C_4^{(7)} & -C_4^{(6)} \\
0 & \frac{P_{03}+P_{14}}{5} & \frac{P_{00}+2P_{11}}{5} & \frac{P_{01}+3P_{12}}{5} & \frac{P_{00}+4P_{11}}{5} & P_{10} & -C_5^{(9)} & -C_5^{(8)} & -C_5^{(7)} & -C_5^{(6)} \\
0 & 0 & \frac{P_{03}+2P_{11}}{6} & \frac{P_{02}+3P_{13}}{6} & \frac{P_{01}+4P_{12}}{6} & \frac{P_{00}+5P_{11}}{6} & -C_6^{(9)} & -C_6^{(8)} & -C_6^{(7)} & -C_6^{(6)} \\
0 & 0 & 0 & \frac{P_{03}+3P_{14}}{7} & \frac{P_{02}+4P_{13}}{7} & \frac{P_{01}+5P_{12}}{7} & -C_7^{(9)} & -C_7^{(8)} & -C_7^{(7)} & 0 \\
0 & 0 & 0 & 0 & \frac{P_{03}+4P_{14}}{8} & \frac{P_{02}+5P_{13}}{8} & -C_8^{(9)} & -C_8^{(8)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{P_{03}+5P_{14}}{9} & -C_9^{(9)} & 0 & 0 & 0
\end{pmatrix}$$

(2.7)

$$\tau = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} \text{ and } b = \begin{pmatrix} P_{10}\alpha_0 \\ f_0 \\ \frac{f_1}{2} \\ \frac{f_2}{3} \\ \frac{f_3}{4} \\ \frac{f_4}{5} \\ \frac{f_5}{6} \\ 0 \\ 0 \end{pmatrix} \qquad (2.8)$$

From (1.1) , the general case for $m = 2$, $s = 2$ is given by

$$\int_0^x \int_0^u (P_{20} + P_{21}t + P_{22}t^2 + P_{23}t^3 + P_{24}t^4) y_n''(t)dtdu + \int_0^x \int_0^u (P_{10} + P_{11}t + P_{12}t^2 + P_{13}t^3) y_n'(t)dtdu + \int_0^x \int_0^u (P_{00} + P_{01}t + P_{02}t^2) y_n(t)dtdu = \int_0^x \int_0^u (\textstyle\sum_{i=0}^n f_i t^i)\, dtdu$$

$$+ \tau_1 T_{n+4}(x) + \tau_2 T_{n+3}(x) + \tau_3 T_{n+2}(x) + \tau_1 T_{n+1}(x) \qquad (2.9)$$

Integrating each terms (2.9) ,collecting the like terms and equating the corresponding coefficients of x for $n = 5$ we have the tau system $B^*D = G$, where

B =

$$\begin{bmatrix} P_{20} & 0 & 0 & 0 & 0 & 0 & -C_0^{(9)} & -C_0^{(8)} & -C_0^{(7)} & -C_0^{(6)} \\ N_{21} & P_{20} & 0 & 0 & 0 & 0 & -C_1^{(9)} & -C_1^{(8)} & -C_1^{(7)} & -C_1^{(6)} \\ N_{31} & N_{32} & P_{20} & 0 & 0 & 0 & -C_2^{(9)} & -C_2^{(8)} & -C_2^{(7)} & -C_2^{(6)} \\ N_{41} & N_{42} & N_{43} & P_{20} & 0 & 0 & -C_3^{(9)} & -C_3^{(8)} & -C_3^{(7)} & -C_3^{(6)} \\ N_{51} & N_{52} & N_{53} & N_{54} & P_{20} & 0 & -C_4^{(9)} & -C_4^{(8)} & -C_4^{(7)} & -C_4^{(6)} \\ 0 & N_{62} & N_{63} & N_{64} & N_{65} & P_{20} & -C_5^{(9)} & -C_5^{(8)} & -C_5^{(7)} & -C_5^{(6)} \\ 0 & 0 & N_{73} & N_{74} & N_{75} & N_{76} & -C_6^{(9)} & -C_6^{(8)} & -C_6^{(7)} & -C_6^{(6)} \\ 0 & 0 & 0 & N_{84} & N_{85} & N_{86} & -C_7^{(9)} & -C_7^{(8)} & -C_7^{(7)} & 0 \\ 0 & 0 & 0 & 0 & N_{95} & N_{96} & -C_8^{(9)} & -C_8^{(8)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_{106} & -C_9^{(9)} & 0 & 0 & 0 \end{bmatrix} \qquad (2.10)$$

Where,

$N_{21} = P_{10} - P_{21}$, $N_{31} = \frac{P_{00}}{2}$, $N_{41} = \frac{P_{01}}{6}$, $N_{51} = \frac{P_{02}}{12}$, $N_{32} = \frac{P_{01}}{2}$, $N_{42} = \frac{P_{00}+P_{11}}{6}$
$N_{52} = \frac{P_{01}+P_{12}}{12}$, $N_{62} = \frac{P_{02}+P_{13}}{20}$, $N_{43} = \frac{P_{10}+P_{21}}{3}$, $N_{53} = \frac{P_{00}+2P_{11}+2P_{22}}{12}$,

$N_{63} = \frac{P_{01}+2P_{12}+2P_{23}}{20}$,$N_{73} = \frac{P_{02}+2P_{13}+2P_{24}}{30}$, $N_{54} = \frac{P_{10}+2P_{21}}{4}$, $N_{64} = \frac{P_{00}+3P_{11}+6P_{22}}{20}$,

$N_{74} = \frac{P_{01}+3P_{12}+6P_{23}}{30}$, $N_{84} = \frac{P_{02}+3P_{13}+6P_{24}}{42}$, $N_{65} = \frac{P_{10}+3P_{21}}{5}$, $N_{75} = \frac{P_{00}+4P_{11}+12P_{22}}{30}$,

$N_{85} = \frac{P_{01}+4P_{12}+12P_{23}}{42}$, $N_{95} = \frac{P_{02}+4P_{13}+12P_{24}}{56}$, $N_{76} = \frac{P_{10}+4P_{21}}{6}$, $N_{86} = \frac{P_{00}+5P_{11}+20P_{22}}{42}$,

$N_{96} = \frac{P_{01}+5P_{12}+20P_{23}}{56}$, $N_{106} = \frac{P_{02}+5P_{13}+20P_{24}}{72}$

$$D=\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} P_{20}\alpha_0 \\ P_{20}\alpha_1 + (P_{10} - P_{21})\alpha_0 \\ \frac{f_0}{2} \\ \frac{f_1}{6} \\ \frac{f_2}{12} \\ \frac{f_3}{20} \\ \frac{f_4}{30} \\ \frac{f_5}{42} \\ 0 \\ 0 \end{pmatrix} \tag{2.11}$$

For m =3, s = 1

The general form for m =3, s = 1 from (1.1) is

$$\int_0^x \int_0^u \int_0^t (P_{30} + P_{31}w + P_{32}w^2 + P_{33}w^3 + P_{34}w^4) y_n'''(w)dwdtdu + \int_0^x \int_0^u \int_0^t (P_{20} + P_{21}w + P_{22}w^2 + P_{23}w^3)\, y_n''(w)dwdtdu + \int_0^x \int_0^u \int_0^t (P_{10} + P_{11}w + P_{12}w^2)\, y_n'(w)_{n+1}dwdtdu + \int_0^x \int_0^u \int_0^t (P_{00} + P_{01}w)\, y_n(w)dwdtdu = \int_0^x \int_0^u \int_0^t \sum_{i=0}^{n} f_i x^i + H_{n+m+1}(x) \tag{2.12}$$

where, $H_{n+m+1}(x)$ is given in (2.3).Using the same procedure we have the tau system

B*D = G, where in this case

B =

$$\begin{bmatrix} P_{30} & 0 & 0 & 0 & 0 & 0 & -C_0^{(9)} & -C_0^{(8)} & -C_0^{(7)} & -C_0^{(6)} \\ R_{21} & P_{30} & 0 & 0 & 0 & 0 & -C_1^{(9)} & -C_1^{(8)} & -C_1^{(7)} & -C_1^{(6)} \\ R_{31} & R_{32} & P_{30} & 0 & 0 & 0 & -C_2^{(9)} & -C_2^{(8)} & -C_2^{(7)} & -C_2^{(6)} \\ R_{41} & R_{42} & R_{43} & P_{30} & 0 & 0 & -C_3^{(9)} & -C_3^{(8)} & -C_3^{(7)} & -C_3^{(6)} \\ R_{51} & R_{52} & R_{53} & R_{54} & P_{30} & 0 & -C_4^{(9)} & -C_4^{(8)} & -C_4^{(7)} & -C_4^{(6)} \\ 0 & R_{62} & R_{63} & R_{64} & R_{65} & P_{30} & -C_5^{(9)} & -C_5^{(8)} & -C_5^{(7)} & -C_5^{(6)} \\ 0 & 0 & R_{73} & R_{74} & R_{75} & R_{76} & -C_6^{(9)} & -C_6^{(8)} & -C_6^{(7)} & -C_6^{(6)} \\ 0 & 0 & 0 & R_{84} & R_{85} & R_{86} & -C_7^{(9)} & -C_7^{(8)} & -C_7^{(7)} & 0 \\ 0 & 0 & 0 & 0 & R_{95} & R_{96} & -C_8^{(9)} & -C_8^{(8)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{106} & -C_9^{(9)} & 0 & 0 & 0 \end{bmatrix} \tag{2.13}$$

where,

$R_{21} = P_{20} - 2P_{31}$, $R_{31} = \frac{P_{10} - P_{21} + 2P_{31}}{2}$, $R_{41} = \frac{P_{00}}{6}$, $R_{51} = \frac{P_{01}}{24}$, $R_{32} = \frac{P_{20} - P_{31}}{2}$,

$R_{42} = \frac{P_{10}}{6}$, $R_{52} = \frac{P_{00} + P_{11}}{24}$, $R_{62} = \frac{P_{01} + P_{21}}{60}$, $R_{43} = \frac{P_{20}}{3}$, $R_{53} = \frac{P_{10} + P_{21}}{12}$

$R_{63} = \frac{P_{00} + 2P_{11} + 2P_{22}}{60}$, $R_{73} = \frac{P_{01} + 2P_{12} + 2P_{23}}{120}$, $R_{54} = \frac{P_{20} + P_{31}}{4}$, $R_{64} = \frac{P_{10} + 2P_{21} + 2P_{32}}{20}$

$R_{74} = \frac{P_{00} + 3P_{11} + 6P_{22} + 6P_{33}}{120}$, $R_{84} = \frac{P_{01} + 3P_{12} + 6P_{23} + 6P_{34}}{210}$, $R_{65} \frac{P_{20} + 2P_{31}}{5}$, $R_{75} = \frac{P_{10} + 3P_{21} + 6P_{32}}{30}$

$R_{85} = \frac{P_{00} + 4P_{11} + 12P_{22} + 24P_{33}}{210}$, $R_{95} = \frac{P_{01} + 4P_{12} + 12P_{23} + 24P_{34}}{336}$, $R_{76} = \frac{P_{20} + 3P_{31}}{6}$,

$R_{86} = \frac{P_{10} + 4P_{21} + 12P_{32}}{42}$, $R_{96} = \frac{P_{00} + 5P_{11} + 20P_{22} + 60P_{33}}{336}$, $R_{106} = \frac{P_{01} + 4P_{12} + 20P_{23} + 69P_{34}}{504}$

$$D = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} P_{30}\alpha_0 \\ P_{30}\alpha_1 + (P_{20} - 2P_{31})\alpha_0 \\ \frac{P_{30}}{2}\alpha_2 + \left(\frac{P_{20}}{2} - \frac{P_{31}}{2}\right)\alpha_1 + \left(\frac{P_{10}}{2} - \frac{P_{21}}{2} + P_{32}\right)\alpha_0 \\ \frac{f_0}{6} \\ \frac{f_1}{24} \\ \frac{f_2}{60} \\ \frac{f_3}{120} \\ \frac{f_4}{210} \\ \frac{f_5}{336} \\ 0 \end{pmatrix} \tag{2.14}$$

Continuing with the process for m = 4 , s = 0 using (1.1) we have the following tau system

B =

$$\begin{bmatrix} P_{40} & 0 & 0 & 0 & 0 & 0 & -C_0^{(9)} & -C_0^{(8)} & -C_0^{(7)} & -C_0^{(6)} \\ T_{21} & P_{40} & 0 & 0 & 0 & 0 & -C_1^{(9)} & -C_1^{(8)} & -C_1^{(7)} & -C_1^{(6)} \\ R_{31} & T_{32} & P_{40} & 0 & 0 & 0 & -C_2^{(9)} & -C_2^{(8)} & -C_2^{(7)} & -C_2^{(6)} \\ T_{41} & T_{42} & T_{43} & P_{40} & 0 & 0 & -C_3^{(9)} & -C_3^{(8)} & -C_3^{(7)} & -C_3^{(6)} \\ T_{51} & T_{52} & T_{53} & T_{54} & P_{40} & 0 & -C_4^{(9)} & -C_4^{(8)} & -C_4^{(7)} & -C_4^{(6)} \\ 0 & T_{62} & T_{63} & T_{64} & T_{65} & P_{40} & -C_5^{(9)} & -C_5^{(8)} & -C_5^{(7)} & -C_5^{(6)} \\ 0 & 0 & T_{73} & T_{74} & T_{75} & T_{76} & -C_6^{(9)} & -C_6^{(8)} & -C_6^{(7)} & -C_6^{(6)} \\ 0 & 0 & 0 & T_{84} & T_{85} & T_{86} & -C_7^{(9)} & -C_7^{(8)} & -C_7^{(7)} & 0 \\ 0 & 0 & 0 & 0 & T_{95} & T_{96} & -C_8^{(9)} & -C_8^{(8)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_{106} & -C_9^{(9)} & 0 & 0 & 0 \end{bmatrix} \tag{2.15}$$

$$D = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix} \text{ and } G = \begin{bmatrix} P_{40} \\ P_{40}\alpha_1 - 3P_{41}\alpha_0 + P_{30}\alpha_0 \\ \frac{P_{40}}{2}\beta_2 - \frac{2P_{41}}{2}\alpha_1 + \frac{6P_{42}}{2}\alpha_0 + \frac{P_{30}}{2}\alpha_1 + \frac{2P_{31}}{2}\alpha_0 + \frac{P_{20}}{2}\alpha_0 \\ \frac{P_{40}}{6}\alpha_3 - \frac{P_{41}}{6}\alpha_2 + \frac{2P_{42}}{6}\alpha_1 - \frac{6P_{43}}{6}\alpha_0 + \frac{P_{31}}{6}\alpha_2 - \frac{P_{31}}{6}\alpha_1 + \frac{2P_{32}}{6}\alpha_0 \\ \frac{P_{20}}{6}\alpha_1 - \frac{P_{21}}{6}\alpha_0 + \frac{P_{10}}{6}\alpha_0 \\ \frac{f_0}{24} \\ \frac{f_1}{120} \\ \frac{f_2}{360} \\ \frac{f_3}{840} \\ \frac{f_4}{1680} \\ \frac{f_5}{3024} \end{bmatrix} \tag{2.16}$$

where,

$$T_{21} = P_{30} - 3P_{41}, T_{32} = \frac{P_{30} - 2P_{41}}{2}, T_{43} = \frac{P_{30} - P_{41}}{3}, T_{54} = \frac{P_{30}}{4}, T_{65} = \frac{P_{30} + P_{41}}{5},$$

$$T_{76} = \frac{P_{30} + 2P_{41}}{6}, T_{31} = \frac{P_{20} - 2P_{31} + 6P_{43}}{2}, T_{42} = \frac{P_{20} - 2P_{31} + 2P_{42}}{6}, T_{53} = \frac{P_{20}}{12},$$

$$T_{64} = \frac{P_{20} + P_{31}}{20}, T_{75} = \frac{P_{20} + 2P_{31} + 2P_{42}}{30}, T_{86} = \frac{P_{20} + 2P_{31} + 2P_{42}}{42},$$

$$T_{41} = \frac{P_{10} + P_{21} + 2P_{32} - 6P_{43}}{6}, T_{52} = \frac{P_{10}}{24}, T_{63} = \frac{P_{10} + P_{21}}{60}, T_{74} = \frac{P_{10} + 2P_{21} + 2P_{32}}{120},$$

$$T_{85} = \frac{P_{10} + 3P_{21} + 6P_{32} + 6P_{43}}{210}, T_{96} = \frac{P_{10} + 24 + 12P_{32} + 24P_{43}}{386} T_{51} = \frac{P_{00}}{24},$$

$$T_{62} = \frac{P_{00} + P_{11}}{120}, T_{73} = \frac{P_{00} + 2P_{11} + 2P_{22}}{360}, T_{84} = \frac{P_{00} + 3P_{11} + 6P_{22} + 6P_{33}}{840}, T_{53} = \frac{P_{20}}{12},$$

$$T_{95} = \frac{P_{00} + 4P_{11} + 12P_{22} + 24P_{33} + 24P_{44}}{1680}, T_{106} = \frac{P_{00} + 5P_{11} + 20P_{22} + 60P_{33} + 120P_{44}}{3024}$$

We obtained the following expressions for $a'_{i,j}s$ and $b'_i s$ i.e

$$a_{kk} = P_{m0}, \quad \forall k = 1(1)(n+1), \forall m$$

$$a_{kr} = \begin{cases} \frac{\sum_{k=0}^{1} k!\binom{r-m}{k} P_{m-2+k,k}}{r_{p_1}}, & \forall k = 2(1)(n+2), \forall r = 1(1)(n+1), \quad \forall m = 1,2,3\,4. \\ \frac{\sum_{k=0}^{2} k!\binom{r-m+1}{k} P_{m-2+k,k}}{(r+1)_{p_2}}, & \forall k = 3(1)(n+3), \forall r = 1(1)(n+1), \forall m = 2,3,4. \\ \frac{\sum_{k=0}^{3} k!\binom{r-m+2}{k} P_{m-3+k,k}}{(r+2)_{p_3}}, & \forall k = 4(1)(n+4), \forall r = 1(1)(n+1), \quad \forall m = 3,4. \\ \frac{\sum_{k=0}^{4} k!\binom{r-m+3}{k} P_{m-4+k,k}}{(r+3)_{p_4}}, & \forall k = 5(1)(n+5), \forall r = 1(1)(n+1), \quad \forall m = 4 \\ \vdots & \\ \frac{\sum_{k=0}^{m} k!\binom{r-1}{k} P_{kk}}{(r+m-1)_{p_m}}, & \forall k = (m+1)(1)(n+m+1), \quad r = 1(1)(n+1), \quad \forall m \\ \frac{\sum_{k=0}^{m} k!\binom{r-1}{k} P_{k,k+1}}{(r+m)_{p_m}}, & \forall k = (m+s+1)(1)(n+m+2), \quad r = 1(1)(n+1), \forall m \\ \frac{\sum_{k=0}^{m} k!\binom{r-1}{k} P_{k,k+2}}{(r+m+1)_{p_m}}, & \forall k = (m+s+1)(1)(n+m+s), r = 1(1)(n+1), \quad \forall m \\ \vdots & \\ \frac{\sum_{k=0}^{m} k!\binom{r-1}{k} P_{k,k+s}}{(r+m+s-1)_{p_m}}, & \forall k = (m+s+1)(1)(n+m+s), \forall r = 1(1)(n+1), \quad \forall m \end{cases}$$

(2.17a)

and

$$a_{kk} = \begin{cases} 0, & \forall r > k,\ k = 1(1)(n),\ r = 2(1)(n+1) \\ \vdots & \\ 0, & \forall\ (m+s+2)(1)(n+m+s+1)\ , r = 1(1)n \end{cases}$$

$$a_{k,n+2} = -C_{k-1}^{(n+m+s)}, \quad k = 1(1)(n+m+s+1)$$

$$a_{k,n+3} = -C_{k-1}^{(n+m+s-1)}, \quad k = 1(1)(n+m+s)$$

$$a_{k,n+4} = -C_{k-1}^{(n+m+s-2)}, \quad k = 1(1)(n+m+s-1)$$

$$\vdots \quad \vdots$$

$$a_{k,n+m+s+1} = -C_{k-1}^{(n+s-1)}, \quad k = 1(1)(n+s) \qquad (2.17b)$$

and

$b_1 = P_{m0}$, $\forall m$

$$b_2 = \frac{1}{(m-1)!}\{\alpha_1 \sum_{r=0}^{0}(-1)^r r!\,P_{r+2,r} + \alpha_0 \sum_{r=0}^{1}(-1)^r r!\,P_{r+1,r}\}\forall\ m = 2$$

$$b_3 = \frac{1}{(m-1)!}\{\alpha_2 \sum_{r=0}^{0}(-1)^r r!\,P_{r+3,r} + \alpha_1 \sum_{r=0}^{1}(-1)^r r!\,P_{r+2,r} + \alpha_0 \sum_{r=0}^{1}(-1)^r r!\,P_{r+1,r}\}\forall\ m = 3$$

$$b_4 = \frac{1}{(m-1)!}\{\alpha_3 \sum_{r=0}^{0}(-1)^r r!\,P_{r+4,r} + \alpha_2 \sum_{r=0}^{1}(-1)^r r!\,P_{r+3,r} + \alpha_1 \sum_{r=0}^{2}(-1)^r r!\,P_{r+2,r} +$$

$$\alpha_0 \sum_{r=0}^{1}(-1)^r r!\,P_{r+1,r}\}\forall\ m = 4$$

$$b_4 = \frac{P_{40}}{6}\beta_3 + \frac{1}{6}(P_{30} - P_{41})\beta_2 + \frac{1}{6}(P_{20}+P_{31} - 2P_{42})\beta_1 + \frac{1}{6}(P_{10}-P_{21} + 2P_{32} - 6P_{43})\beta_0$$

$$b_i = \frac{f_{i-m-1}}{\prod_{r=1}^{m}(i-r)}\ ,\quad \forall\quad m+1 \le i \le 2n-s\ ,\ 1 \le m \le 4 \qquad (2.17c)$$

Analysis of error estimation

The integrated formulation of the tau method often yields to better accuracy of the tau solution .To this end we consider the perturbed form of (1.14), ie perturbed error equation

$$I_L(e_n(x))_{n+1} = \iiint \dots \int H_n(x)dx\ \ H_{n+m+1}(x) \qquad (3.1)$$

which is satisfied by by$(e_n(x))_{n+1}$ given by

$$(e_n(x))_{n+1} = \frac{\phi_n \mu_m(x) T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \qquad (3.2)$$

withϕ_n replaced with by $\hat{\phi}_n$ and $\bar{H}_{n+m+1}(x)$

where ,

$$\bar{H}_{n+m+1}(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-1}\ T_{n-m+r+3}(x) \qquad (3.3)$$

Equating the corresponding coefficients of like powers of x in (3.1) and solving the resulting algebraic equations lead to the value of φ_n, we then have

$$\varepsilon = \max_{0\le x\le b} |\ (e_n(x))_{n+1}\ | = \frac{|\hat{\phi}_n|}{|C_{n-m+1}^{(n-m+1)}|} \simeq \max_{a\le x\le b} |(e_n(x))| \qquad (3.4)$$

as an estimate of ε.

We shall carry out these steps for obtaining with various values of m +s = 4 and then generate the result to obtain the recursive formula for ϕ_n.

The case m =1, s = 3

From (1.1), the most general case for m =1 and s = 3 is given by

$$\int_0^x (P_{10} + P_{11}t + P_{12}t^2 + P_{13}t^3 + P_{14}t^4)\,(e_n'(x))_{n+1}dt + \int_0^x (P_{00} + P_{01}t + P_{02}t^2 +$$

$P_{03}t^3)(e_n(x))_{n+1}dt = -\int_0^x H_n(t)\,dt + \bar{H}_{n+m+1}(x)$ (3.5)

where ,

$$(e_n(x))_{n+1} = \frac{\phi_n T_n(x)}{C_n^{(n)}} \quad (3.6)$$

that is,

$$(e_n(x))_{n+1} = \frac{\phi_n}{k_1}\{k_1x^{n+1} + k_2x^n + k_3x^{n-1} + + + \cdots\} \quad (3.7)$$

where,

$k_1 = C_n^{(n)}$, $k_2 = C_{n-1}^{(n)}$, $k_2 = C_{n-2}^{(n)}$ e.tc.

Thus ,

$$\int_0^x (e_n(t))_{n+1}\,dt = \frac{\phi_n}{k_1}\left\{\frac{k_1x^{n+2}}{n+2} + \frac{k_2x^{n+1}}{n+1} + \frac{k_3x^n}{n} + + + \cdots\right\} \quad (3.8)$$

Inserting (3.7) and (3.8) into (3,5) yields

$$\frac{\phi_n}{k_1}\{\lambda_1x^{n+5} + \lambda_2x^{n+4} + \lambda_3x^{n+3} + \lambda_4x^{n+2} + \lambda_5x^{n+1} + \cdots\} = \hat{\tau}_1C_{n+5}^{(n+5)}x^{n+5} +$$

$$\{\hat{\tau}_1C_{n+4}^{(n+5)} + \hat{\tau}_2C_{n+4}^{(n+4)} - \frac{\tau_1C_{n+3}^{(n+3)}}{n+4}\}x^{n+4} + \{\hat{\tau}_1C_{n+3}^{(n+5)} + \hat{\tau}_2C_{n+3}^{(n+4)} + \hat{\tau}_3C_{n+3}^{(n+3)} - \frac{\tau_1C_{n+2}^{(n+3)}}{n+3} -$$

$$\frac{\tau_2C_{n+2}^{(n+2)}}{n+3}\}x^{n+3} + \{\hat{\tau}_1C_{n+2}^{(n+5)} + \hat{\tau}_2C_{n+2}^{(n+4)} + \hat{\tau}_3C_{n+2}^{(n+3)} + \hat{\tau}_4C_{n+2}^{(n+2)} - \frac{\tau_1C_{n+2}^{(n+3)}}{n+2} - \frac{\tau_2C_{n+2}^{(n+2)}}{n+2} -$$

$$\frac{\tau_3C_{n+1}^{(n+1)}}{n+2}\}x^{n+2} + \{\hat{\tau}_1C_{n+1}^{(n+5)} + \hat{\tau}_2C_{n+1}^{(n+4)} + \hat{\tau}_3C_{n+1}^{(n+3)} + \hat{\tau}_4C_{n+1}^{(n+2)} - \frac{\tau_1C_n^{(n+3)}}{n+1} - \frac{\tau_2C_n^{(n+2)}}{n+1} -$$

$$\frac{\tau_3C_n^{(n+1)}}{n+1} - \frac{\tau_3C_n^{(n)}}{n+1}\}x^{n+1} + + --- \quad (3.9)$$

Equating coefficients of corresponding powers of x from both sides of (3.9) gives

$$\hat{\tau}_1C_{n+5}^{(n+5)} = \frac{\varphi_n\lambda_1}{k_1} \quad (3.10a)$$

$$\hat{\tau}_1C_{n+4}^{(n+5)} + \hat{\tau}_2C_{n+4}^{(n+4)} - \frac{\tau_1C_{n+3}^{(n+3)}}{n+4} = \frac{\varphi_n\lambda_2}{k_1} \quad (3.10b)$$

$$\hat{\tau}_1C_{n+3}^{(n+5)} + \hat{\tau}_2C_{n+3}^{(n+4)} + \hat{\tau}_3C_{n+3}^{(n+3)} - \frac{\tau_1C_{n+2}^{(n+3)}}{n+3} - \frac{\tau_2C_{n+2}^{(n+2)}}{n+3} = \frac{\varphi_n\lambda_3}{k_1} \quad (3.10c)$$

$$\hat{\tau}_1C_{n+2}^{(n+5)} + \hat{\tau}_2C_{n+2}^{(n+4)} + \hat{\tau}_3C_{n+2}^{(n+3)} + \hat{\tau}_4C_{n+2}^{(n+2)} - \frac{\tau_1C_{n+2}^{(n+3)}}{n+2} - \frac{\tau_2C_{n+2}^{(n+2)}}{n+2} - \frac{\tau_3C_{n+1}^{(n+1)}}{n+2} = \frac{\varphi_n\lambda_4}{k_1} \quad (3.10d)$$

$$\hat{\tau}_1 C_{n+1}^{(n+5)} + \hat{\tau}_2 C_{n+1}^{(n+4)} + \hat{\tau}_3 C_{n+1}^{(n+3)} + \hat{\tau}_4 C_{n+1}^{(n+2)} - \frac{\tau_1 C_n^{(n+3)}}{n+1} \frac{\tau_2 C_n^{(n+2)}}{n+1} - \frac{\tau_3 C_n^{(n+1)}}{n+1} - \frac{\tau_4 C_n^{(n)}}{n+1} = \frac{\varphi_n \lambda_5}{k_1} \tag{3.10e}$$

Using well – known relations.

$$C_n^{(n)} = 2^{2n-1}, C_{n-1}^{(n)} = \frac{-1}{2} n C_n^{(n)}, C_{n-1}^{(n)} = -n2^{2n-2}$$

We solve this system of equations (3.10a) – (3.10e) by forward substitution for $\phi_n toobtain$

$$\phi_n = -\{\frac{C_{n+2}^{(n+4)} C_{n+3}^{(n+3)} C_{n+1}^{(n+2)}}{(n+4) C_{n+4}^{(n+4)} C_{n+2}^{(n+2)}} - \frac{C_{n+1}^{(n+4)} C_{n+3}^{(n+3)}}{(n+4) C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+3)} C_{n+1}^{(n+2)}}{(n+2) C_{n+2}^{(n+2)}} + \frac{C_n^{(n+3)}}{(n+1)}\} \frac{K_1 \tau_1}{R_5} + \{\frac{C_{n+1}^{(n+3)} C_{n+2}^{(n+2)}}{(n+3) C_{n+3}^{(n+3)}} - \frac{C_n^{(n+2)}}{(n+1)}\} \frac{K_1 \tau_2}{R_5}$$

$$\frac{K_1^2 \tau_4}{(n+1) R_5} \tag{3.11}$$

where ,

$$R_5 = \lambda_5 - \frac{C_{n+1}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+1}^{(n+4)} R_2}{C_{n+4}^{(n+4)}} \frac{C_{n+1}^{(n+3)} R_3}{C_{n+3}^{(n+3)}} - \frac{C_{n+1}^{(n+2)} R_4}{C_{n+2}^{(n+2)}}$$

$$R_4 = \lambda_4 - \frac{C_{n+2}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+2}^{(n+4)} R_2}{C_{n+4}^{(n+4)}} \frac{C_{n+2}^{(n+3)} R_3}{C_{n+3}^{(n+3)}}$$

$$R_3 = \lambda_3 - \frac{C_{n+3}^{(n+5)} R_1}{C_{n+5}^{(n+5)}} - \frac{C_{n+3}^{(n+4)} R_2}{C_{n+4}^{(n+4)}}, R_2 = \lambda_2 - \frac{C_{n+4}^{(n+5)} R_1}{C_{n+5}^{(n+5)}}, R_1 = \lambda_1 \tag{3.12a}$$

and

$$\lambda_1 = \left[\frac{P_{03} + (n+1) P_{14}}{(n+5)}\right] K_1 \quad , \quad \lambda_2 = \left[\frac{P_{02} + (n+1) P_{13}}{(n+4)}\right] K_1 + \left[\frac{P_{03} + n P_{14}}{(n+4)}\right] K_2 \ ,$$

$$\lambda_3 = \left[\frac{P_{01} + (n+1) P_{12}}{(n+3)}\right] K_1 + \left[\frac{P_{02} + n P_{13}}{(n+3)}\right] K_2 + \left[\frac{P_{03} + (n-1) P_{14}}{(n+3)}\right] K_3$$

$$\lambda_4 = \left[\frac{P_{00} + (n+1) P_{11}}{(n+2)}\right] K_1 \left[\frac{P_{01} + n P_{12}}{(n+2)}\right] K_2 + \left[\frac{P_{02} + (n-1) P_{13}}{(n+2)}\right] K_3 + \left[\frac{P_{03} + (n-2) P_{14}}{(n+2)}\right] K_4$$

$$\lambda_5 = P_{10} K_1 + \left[\frac{P_{00} + n P_{11}}{(n+1)}\right] K_2 + \left[\frac{P_{01} + (n-1) P_{12}}{(n+1)}\right] K_3 + \left[\frac{P_{02} + (n-2) P_{13}}{(n+1)}\right] K_4 + \left[\frac{P_{03} + (n-3) P_{14}}{(n+1)}\right] K_5 \tag{3.12b}$$

For m = 2, s = 2

In this case we have from (1.1)

$$\int_0^x \int_0^u (P_{20} + P_{21} t + P_{22} t^2 + P_{23} t^3 + P_{24} t^4)(e_n''(t))_{n+1} dt du + \int_0^x \int_0^u (\int_0^x \int_0^u (P_{10} + P_{11} t +$$

$$P_{12} t^2 + P_{13} t^3)(e_n'(t))_{n+1} dt du + \int_0^x \int_0^u (P_{00} + P_{01} t + P_{02} t^2)(e_n(t))_{n+1} dt du =$$

$$- \int_0^x \int_0^u (H_n(t) dt du \ + \ H_{m+n+1}(x) \tag{3.13}$$

$$(e_n(x))_{n+1} = \frac{\phi_n (x-\alpha)^m T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{\phi_n x^2 T_{n-1}(x)}{C_{n-1}^{(n-1)}} = \frac{\phi_n x^2 T_{n-1}(x)}{2^{2n-3}} \tag{3.14}$$

that is,

$$(e_n(x))_{n+1} = \frac{\phi_n}{k_1}\{k_1x^{n+1} + k_2x^n + k_3x^{n-1} + + + \cdots\} \quad (3.15a)$$

and ,

$$\int_0^x\int_0^u((e_n(t))_{n+1}dtdu = \frac{\phi_n}{k_1}\left\{\frac{k_1x^{n+3}}{(n+2)(n+3)} + \frac{k_2x^{n+2}}{(n+1)(n+2)} + \frac{k_3x^{n+1}}{n(n+1)} + + + \cdots\right\} \quad (3.15b)$$

$$\int_0^x\int_0^u((e_n''(t))_{n+1}dtdu = (e_n(x))_{n+1} = \frac{\phi_n}{K_1}\{k_1x^{n+1} + k_2x^n + k_3x^{n-1} + + \cdots\} \quad (3.15c)$$

$$\int_0^x\int_0^u((e_n'(t))_{n+1}dtdu = \frac{\phi_n}{k_1}\left\{\frac{k_1x^{n+2}}{(n+2)} + \frac{k_2x^{n+1}}{(n+1)} + \frac{k_3x^n}{n} + + + \cdots\right\} \quad (3.15d)$$

Substituting (3.14), (3.15a) – (3.15d), collecting the like terms and equating the corresponding coefficients of $x^{n+5}, x^{n+4}, x^{n+3}, x^{n+2} and x^{n+1}$, we obtain the following system of equations.

$$\hat{\tau}_1C_{n+5}^{(n+5)} = \frac{\phi_n\lambda_1}{k_1} \quad (3.16a)$$

$$\hat{\tau}_1C_{n+4}^{(n+5)} + \hat{\tau}_2C_{n+4}^{(n+4)} - \frac{\tau_1C_{n+2}^{(n+2)}}{(n+3)(n+4)} = \frac{\phi_n\lambda_2}{k_1} \quad (3.16b)$$

$$\hat{\tau}_1C_{n+3}^{(n+5)} + \hat{\tau}_2C_{n+3}^{(n+4)} + \hat{\tau}_3C_{n+3}^{(n+3)} - \frac{\tau_1C_{n+1}^{(n+2)}}{(n+2)(n+3)} - \frac{\tau_2C_{n+1}^{(n+1)}}{(n+2)(n+3)} = \frac{\phi_n\lambda_3}{k_1} \quad (3.16c)$$

$$\hat{\tau}_1C_{n+2}^{(n+5)} + \hat{\tau}_2C_{n+2}^{(n+4)} + \hat{\tau}_3C_{n+2}^{(n+3)} + \hat{\tau}_4C_{n+2}^{(n+2)} - \frac{\tau_1C_n^{(n+2)}}{(n+1)(n+2)} - \frac{\tau_2C_n^{(n+1)}}{(n+1)(n+2)} - \frac{\tau_3C_n^{(n)}}{(n+1)(n+2)} = \frac{\phi_n\lambda_4}{k_1} \quad (3.16d)$$

$$\hat{\tau}_1C_{n+1}^{(n+5)} + \hat{\tau}_2C_{n+1}^{(n+4)} + \hat{\tau}_3C_{n+1}^{(n+3)} + \hat{\tau}_4C_{n+1}^{(n+2)} - \frac{\tau_1C_{n-1}^{(n+2)}}{n(n+1)} - \frac{\tau_2C_{n-1}^{(n+1)}}{n(n+1)} - \frac{\tau_3C_{n-1}^{(n)}}{n(n+1)} - \frac{\tau_4C_{n-1}^{(n-1)}}{n(n+1)} = \frac{\phi_n\lambda_5}{k_1}$$

where,

$$\lambda_1 = \left[\frac{P_{02}+(n+1)P_{13}+n(n+1)P_{24}}{(n+4)(n+5)}\right]K_1 \quad , \quad \lambda_2 = \left[\frac{P_{01}+(n+1)P_{12}+n(n+1)P_{23}}{(n+3)(n+4)}\right]K_1 + \left[\frac{P_{02}+nP_{13}+n(n-1)P_{24}}{(n+3)(n+4)}\right]K_2$$

$$\lambda_3 = \left[\frac{P_{00}+(n+1)P_{11}+n(n+1)P_{22}}{(n+2)(n+3)}\right]K_1 + \left[\frac{P_{01}+nP_{12}+n(n-1)P_{23}}{(n+2)(n+3)}\right]K_2 + \left[\frac{P_{02}+(n-1)P_{13}+(n-1)(n-2)P_{24}}{(n+2)(n+3)}\right]K_3$$

$$\lambda_4 = \left[\frac{P_{10}+nP_{21}}{(n+2)}\right]K_1 + \left[\frac{P_{00}+nP_{11}+n(n-1)P_{13}}{(n+1)(n+2)}\right]K_2 + \left[\frac{P_{01}+(n-1)P_{12}+(n-1)(n-2)P_{23}}{(n+1)(n+2)}\right]K_3 + \left[\frac{P_{02}+(n-2)P_{13}+(n-2)(n-3)P_{24}}{(n+1)(n+2)}\right]K_4$$

$$\lambda_5 = P_{20}K_1 + \left[\frac{P_{10}+(n-1)P_{21}}{(n+1)}\right]K_2 + \left[\frac{P_{00}+(n-1)P_{11}+(n-1)(n-2)P_{22}}{n(n+1)}\right]K_3 + \left[\frac{P_{01}+(n-2)P_{12}+(n-2)(n-3)P_{23}}{n(n+1)}\right]K_4 + \left[\frac{P_{02}+(n-3)P_{13}+(n-3)(n-4)P_{24}}{(n+1)}\right]K_5 \quad (3.17)$$

$k_1 = C_{n-1}^{(n-1)}$, $k_2 = C_{n-2}^{(n-1)}$, $k_3 = C_{n-3}^{(n-1)}$, $k_4 = C_{n-4}^{(n-1)}$ e.t.c

Following the same procedure the solution of system of equations (3.16a)- (3.16e) yields,

$$\phi_n = -\{\frac{C_{n+2}^{(n+4)}C_{n+2}^{(n+2)}C_{n+1}^{(n+2)}}{(n+3)(n+4)C_{n+4}^{(n+4)}C_{n+2}^{(n+2)}} - \frac{C_{n+1}^{(n+4)}C_{n+2}^{(n+2)}}{(n+3)(n+4)C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+2)}C_{n}^{(n+2)}}{(n+1)(n+2)C_{n+2}^{(n+2)}} + \frac{C_{n-1}^{(n+2)}}{n(n+1)}\}\frac{K_1\tau_1}{R_5}$$

$$+\{\frac{C_{n+1}^{(n+3)}C_{n+1}^{(n+1)}}{(n+2)(n+3)C_{n+3}^{(n+3)}} \quad \frac{C_{n-1}^{(n+1)}}{n(n+1)}\}\frac{K_1\tau_2}{R_5} \quad \frac{K_1^2\tau_4}{n(n+1)R_5} \qquad (3.18)$$

where recursive expression for R is as obtained in (3.12a)

For m =3, s = 1

The general form for m =3, s = from (1.1) is

$$\int_0^x\int_0^u\int_0^t(P_{30} + P_{31}w + P_{32}w^2 + P_{33}w^3 + P_{34}w^4)(e_n'''(w))_{n+1}dwdtdu + \int_0^x\int_0^u\int_0^t(P_{20} + P_{21}w +$$

$$P_{22}w^2 + P_{23}w^3)\,(e_n''(w))_{n+1}dwdtdu + \int_0^x\int_0^u\int_0^t(P_{10} + P_{11}w + P_{12}w^2)\,(e_n'(w))_{n+1}dwdtdu +$$

$$\int_0^x\int_0^u\int_0^t(P_{00} + P_{01}w)\,(e_n(w))_{n+1}dwdtdu = -\int_0^x\int_0^u\int_0^t H_n(w)dwdtdu + \tilde{H}_{n+m+1}(x) \qquad (3.19)$$

where,

$$(e_n(x))_{n+1} = \frac{\phi_n(x-\alpha)^m T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{\phi_n x^3 T_{n-2}(x)}{C_{n-2}^{(n-2)}} = \frac{\phi_n x^3 T_{n-2}(x)}{2^{2n-5}}$$

$$H_n(x) = \tau_1 T_{n+1}(x) + \tau_2 T_n(x) + \tau_3 T_{n-1}(x) + \tau_4 T_{n-2}(x)$$

$$\tilde{H}_{n+m+1} = \bar{\tau}_1 T_{n+5}^*(x) + \bar{\tau}_2 T_{n+4}^*(x) + \bar{\tau}_3 T_{n+3}^*(x) + \bar{\tau}_4 T_{n+2}^*(x) \qquad (3.20)$$

After substituting (3.20) into (3.19) and solving $\phi_n as$ the resulting equations we obtained

$$\phi_n = -\{\frac{C_{n+2}^{(n+4)}C_{n+1}^{(n+1)}C_{n+1}^{(n+2)}}{(n+2)(n+3)(n+4)C_{n+4}^{(n+4)}C_{n+2}^{(n+2)}} - \frac{C_{n+1}^{(n+4)}C_{n+1}^{(n+1)}}{(n+2)(n+3)(n+4)C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+2)}C_{n-1}^{(n+1)}}{n(n+1)(n+2)C_{n+2}^{(n+2)}}$$

$$+\frac{C_{n-2}^{(n+1)}}{(n-1)n(n+1)}\}\frac{K_1\tau_1}{R_5}$$

$$+\{\frac{C_{n+1}^{(n+3)}C_{n}^{(n)}}{(n+1)(n+2)(n+3)C_{n+3}^{(n+3)}} \quad \frac{C_{n-2}^{(n)}}{(n-1)n(n+1)}\}\frac{K_1\tau_2}{R_5} \quad \frac{K_1^2\tau_4}{(n-1)n(n+1)R_5} \qquad (3.21)$$

where R is obtained recursively as in (3.12a) and

$$\lambda_1 = \left[\frac{P_{01} + (n+1)P_{12} + n(n+1)P_{23} + (n+1)n(n-1)P_{34}}{(n+3)(n+4)(n+5)}\right]K_1$$

$$\lambda_2 = \left[\frac{P_{00}+(n+1)P_{11} + (n+1)nP_{22}+(n+1)n(n-1)P_{33}}{(n+2)(n+3)(n+4)}\right]K_1 + \left[\frac{P_{01}+nP_{12}+n(n-1)P_{23}+n(n-1)(n-2)P_{34}}{(n+2)(n+3)(n+4)}\right]K_2$$

$$\lambda_3 = \left[\frac{P_{10}+nP_{21}+n(n+1)P_{32}}{(n+2)(n+3)}\right]K_1+\left[\frac{P_{00}+nP_{11}+n(n-1)P_{22}+n(n-1)(n-2)P_{33}}{(n+1)(n+2)(n+3)}\right]K_2+$$
$$\left[\frac{P_{01}+(n-1)P_{12}+(n-1)(n-2)P_{23}+(n-1)(n-2)(n-3)P_{34}}{(n+1)(n+2)(n+3)}\right]K_3+$$

$$\lambda_4 = \left[\frac{P_{20}+(n-1)P_{31}}{(n+2)}\right]K_1+\left[\frac{P_{10}+(n-1)P_{21}+(n-1)(n-2)P_{32}}{(n+1)(n+2)}\right]K_2+$$
$$\left[\frac{P_{00}+(n-1)P_{11}+(n-1)(n-2)P_{22}+(n-1)(n-2)(n-3)P_{33}+}{n(n+1)(n+2)}\right]K_3+$$
$$\left[\frac{P_{01}+(n-2)P_{12}+(n-2)(n-3)P_{23}+(n-2)(n-3)(n-4)P_{34}+}{n(n+1)(n+2)}\right]K_4+$$

$$\lambda_5 = P_{30}K_1+\left[\frac{P_{20}+(n-2)P_{31}}{(n+1)}\right]K_2+\left[\frac{P_{10}+(n-2)P_{21}+(n-2)(n-3)P_{32}}{n(n+1)}\right]K_3+$$
$$\left[\frac{P_{00}+(n-2)P_{11}+(n-2)(n-3)P_{22}+(n-2)(n-3)(n-4)P_{33}}{(n-1)n(n+1)}\right]K_4+$$
$$\left[\frac{P_{01}+(n-3)P_{12}+(n-3)(n-4)P_{23}+(n-3)(n-4)(n-5)P_{34}}{(n-1)n(n+1)}\right]K_5 \quad (3.22)$$

For m = 4 , s = 0.

From (1.1), we have :

$$\int_0^x\int_0^u\int_0^t\int_0^w (P_{40}+P_{41}v+P_{42}v^2+P_{43}v^3+P_{44}v^4)\,(e_n^{iv}(v))_{n+1}dvdwdtdu+$$

$$\int_0^x\int_0^u\int_0^t\int_0^w (P_{30}+P_{31}v+P_{32}v^2+P_{33}v^3)\,(e_n'''(v))_{n+1}dvdwdtdu+$$

$$\int_0^x\int_0^u\int_0^t\int_0^w (P_{20}+P_{21}v+P_{22}v^2)\,(e_n''(v))_{n+1}dvdwdtdu+$$

$$\int_0^x\int_0^u\int_0^t\int_0^w (P_{10}+P_{11}v)\,(e_n^{iv}(w))_{n+1}dvdwdtdu+\int_0^x\int_0^u\int_0^t\int_0^w P_{00}\,(e_n(v))_{n+1}dvdwdtdu \quad (3.23)$$

where,

$$(e_n(x))_{n+1} = \frac{\phi_n(x-\alpha)^m T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{\phi_n x^4 T_{n-3}(x)}{C_{n-3}^{(n-3)}}$$
$$H_n(x) = \tau_1T_n(x)+\tau_2T_{n-1}(x)+\tau_3T_{n-2}(x)+\tau_4T_{n-3}(x)$$
$$\bar{H}_{n+m+1} = \bar{\tau}_1T^*_{n+5}(x)+\bar{\tau}_2T^*_{n+4}(x)+\bar{\tau}_3T^*_{n+3}(x)+\bar{\tau}_4T^*_{n+2}(x) \quad (3.24)$$

Inserting (3.24) into (3.23) and solving the resulting equations using well-known relations we obtained the value of $\phi_n as$

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$$\phi_n = -\{\frac{C_{n+2}^{(n+4)}C_n^{(n)}C_{n+1}^{(n+2)}}{(n+1)(n+2)(n+3)(n+4)C_{n+4}^{(n+4)}C_{n+2}^{(n+2)}} - \frac{C_{n+1}^{(n+4)}C_n^{(n)}}{(n+1)(n+2)(n+3)(n+4)C_{n+4}^{(n+4)}} - \frac{C_{n+1}^{(n+2)}C_{n-2}^{(n)}}{(n-1)n(n+1)(n+2)C_{n+2}^{(n+2)}} + \frac{C_{n-3}^{(n)}}{(n-2)(n-1)n(n+1)}\}\frac{K_1\tau_1}{R_5} + \{\frac{C_{n+1}^{(n+3)}C_{n-1}^{(n-1)}}{n(n+1)(n+2)(n+3)C_{n+3}^{(n+3)}} \frac{C_{n-3}^{(n-1)}}{(n-2)(n-1)n(n+1)}\}\frac{K_1\tau_2}{R_5} - \frac{K_1^2\tau_4}{(n-2)(n-1)n(n+1)R_5} \qquad (3.25)$$

where R is obtained recursively as in (3.12a) and

$$\lambda_1 = \left[\frac{P_{00} + (n+1)P_{11} + n(n+1)P_{22} + (n+1)n(n-1)P_{33} + (n+1)n(n-1)(n-2)P_{44}}{(n+2)(n+3)(n+4)(n+5)}\right]K_1$$

$$\lambda_2 = \left[\frac{P_{10} + nP_{21} + n(n-1)nP_{32} + n(n-1)(n-2)P_{43}}{(n+2)(n+3)(n+4)}\right]K_1 + \left[\frac{P_{00} + nP_{11} + n(n-1)P_{22} + n(n-1)(n-2)P_{33} + n(n-1)(n-2)(n-3)P_{44}}{(n+1)(n+2)(n+3)(n+4)}\right]K_2$$

$$\lambda_3 = \left[\frac{P_{20} + (n-1)P_{31} + (n-1)(n-2)P_{42}}{(n+2)(n+3)}\right]K_1 + \left[\frac{P_{10} + (n-1)P_{21} + (n-1)(n-2)P_{32} + (n-1)(n-2)(n-3)P_{43}}{(n+1)(n+2)(n+3)}\right]K_2 + \left[\frac{P_{00} + (n-1)P_{11} + (n-1)(n-2)P_{22} + (n-1)(n-2)(n-3)P_{33} + (n-1)(n-2)(n-3)(n-4)P_{44}}{n(n+1)(n+2)(n+3)}\right]K_3$$

$$\lambda_4 = \left[\frac{P_{30} + (n-2)P_{41}}{(n+2)}\right]K_1 + \left[\frac{P_{20} + (n-2)P_{31} + (n-2)(n-3)P_{42}}{(n+1)(n+2)}\right]K_2 + \left[\frac{P_{10} + (n-2)P_{21} + (n-2)(n-3)P_{32} + (n-2)(n-3)(n-4)P_{43} +}{n(n+1)(n+2)}\right]K_3 + \left[\frac{P_{00} + (n-2)P_{11} + (n-2)(n-3)P_{22} + (n-2)(n-3)(n-4)P_{33} + (n-2)(n-3)(n-4)(n-5)P_{44}}{(n-1)n(n+1)(n+2)}\right]K_4 +$$

$$\lambda_5 = P_{40}K_1 + \left[\frac{P_{30} + (n-3)P_{41}}{(n+1)}\right]K_2 + \left[\frac{P_{20} + (n-3)P_{31} + (n-3)(n-4)P_{42}}{n(n+1)}\right]K_3 + \left[\frac{P_{10} + (n-3)P_{21} + (n-3)(n-4)P_{32} + (n-3)(n-4)(n-5)P_{43}}{(n-1)n(n+1)}\right]K_4 + \left[\frac{P_{00} + (n-3)P_{11} + (n-3)(n-4)P_{22} + (n-3)(n-4)(n-5)P_{33} + (n-3)(n-4)(n-5)(n-6)P_{44}}{(n-2)(n-1)n(n+1)}\right]K_5 \qquad (3.26)$$

We observed that the expression for the ϕ_n was the same for the grouping. ie for $m = 1$ $s = 3$, $m = 2$, $s = 2$, $m = 3$, $s = 1$,and $m= 4$,$s = 0$ $(m + s = 4)$. Consequently, from (3.11) , (3.18) , (3.21) and (3.25) the general expression for $\phi_n$is

$$\phi_n = -\{ \frac{C_{n+m+s-2}^{(n+m+s)} C_{n+s}^{(n+s)} C_{n+m+s-3}^{(n+m+s-2)}}{\prod_{r=1}^{m}(n+s+r)\, C_{n+m+s}^{(n+m+s)} C_{n+m+s-2}^{(n+m+s-2)}} - \frac{C_{n+m+s-3}^{(n+m+s)} C_{n+s}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r)\, C_{n+m+s}^{(n+m+s)}} - \frac{C_{n+m+s-3}^{(n+m+s-2)} C_{n+s-2}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r-2)\, C_{n+m+s-2}^{(n+m+s-2)}}$$

$$+ \frac{C_{n+s-3}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r-3)}\} \frac{K_1\tau_1}{R_{m+s+1}} + \{ \frac{C_{n+m+s-3}^{(n+m+s-1)} C_{n+s-1}^{(n+s-1)}}{\prod_{r=1}^{m}(n+s+r-1) C_{n+m+3-1}^{(n+m+3-1)}} \frac{C_{n+s-3}^{(n+s-1)}}{\prod_{r=1}^{m}(n+s+r-3)}\} \frac{K_1\tau_2}{R_{m+s+1}}$$

$$- \frac{K_1^2 \tau_{m+s}}{\prod_{r=1}^{m}(n+s+r-3) R_{m+s+1}} \qquad \forall\, m+s=4. \qquad (3.27)$$

Thus from (3.4), replacing ϕ_n with $\hat{\phi}_n$ have the following expression for ε^*

$$\varepsilon^* = -\{ \frac{C_{n+m+s-2}^{(n+m+s)} C_{n+s}^{(n+s)} C_{n+m+s-3}^{(n+m+s-2)}}{\prod_{r=1}^{m}(n+s+r)\, C_{n+m+s}^{(n+m+s)} C_{n+m+s-2}^{(n+m+s-2)}} - \frac{C_{n+m+s-3}^{(n+m+s)} C_{n+s}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r)\, C_{n+m+s}^{(n+m+s)}} - \frac{C_{n+m+s-3}^{(n+m+s-2)} C_{n+s-2}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r-2)\, C_{n+m+s-2}^{(n+m+s-2)}}$$

$$+ \frac{C_{n+s-3}^{(n+s)}}{\prod_{r=1}^{m}(n+s+r-3)}\} \frac{K_1\tau_1}{R_{m+s+1}} + \{ \frac{C_{n+m+s-3}^{(n+m+s-1)} C_{n+s-1}^{(n+s-1)}}{\prod_{r=1}^{m}(n+s+r-1) C_{n+m+3-1}^{(n+m+3-1)}} \frac{C_{n+s-3}^{(n+s-1)}}{\prod_{r=1}^{m}(n+s+r-3)}\} \frac{K_1\tau_2}{R_{m+s+1}}$$

$$- \frac{K_1^2 \tau_{m+s}}{\prod_{r=1}^{m}(n+s+r-3) R_{m+s+1}} \qquad \forall\, m+s=4 \qquad (3.28)$$

Numerical Examples

In this section, we consider the application of the tau system and general error estimation formula obtained for the class of ordinary differential equations characterized by m +s = 4to some examples. The exact error is defined as

$$\varepsilon *= \max_{0\le x\le 1}\{|y(x_k) - y_n(x_k)|\},$$

$0 \le x \le 1,\, for k = 0(1)100, \{x_k\} = \{0.01k\}$

(4.1)

Example 4.1

A Second Order Linear Homogeneous Variable Coefficient Problem **Fox** (1968)

$y''(x) - 2(1+x^2)y(x) = 0$(4.2a)

$y(0) = 1, y'(x) = 0$(4.2b)

with true solution

$y'(x) = e^{x^2}$, $0 \le x \le 1$.

For this case m = 2 , s = 2. See table 4.1 for numerical results. The numerical results were presented in the tables below the examples.

Table 1: Error and Error Estimates for Example 4.1

Error/ Degree(n	2	3	4	6
ε	2.82×10^{-3}	6.34×10^{-4}	3.38×10^{-5}	1.90×10^{-8}
ε^*	8.57×10^{-2}	8.76×10^{-4}	9.00×10^{-5}	3.83×10^{-7}

Example 4.2

A Fourth Order Non-Homogeneous Constant Coefficient Problem

$$Ly(x) = y^{iv}(x) - 3601y'''(x) + 3600y(x) = -1 + 1800x^2 \qquad (4.3a)$$

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$y(0) = 1, \quad y'(0) = 1, y''(0) = 1, y'''(0) = 1$ (4.3b)

with the exact solution

$y(x) = 1 + \frac{x^2}{2} + \sinh x$ (4.3c)

Table 2: Error and error estimates for example 4.2

Error/Degree(n)	2	3	4	6
ε	3.37×10^{-4}	2.12×10^{-8}	1.11×10^{-10}	4.23×10^{-13}
ε^*	1.37×10^{-3}	2.10×10^{-7}	5.50×10^{-10}	8.60×10^{-12}

Example 4.3

A Fourth Order Homogenous Constant Coefficient problem

$Ly(x) = y^{iv}(x) - 4y(x) = 0, \quad 0 \le x \le 1 (4.4a)$

$y(0) = 1, \quad y'(0) = 0 \quad y''(0) = 2, \quad y'''(0) = 2.$ (4.4b)

with analytic solution

$y(x) = \frac{e^{\sqrt{2}x} + e^{-\sqrt{2}x}}{2}$ (4.4c)

The Numerical results are presented in Table 3 below

Error/Degree(n)	2	3	4	6
ε	5.79×10^{-4}	1.67×10^{-6}	2.70×10^{-8}	9.31×10^{-16}
ε^*	4.93×10^{-4}	1.46×10^{-6}	1.93×10^{-8}	7.61×10^{-10}

Results and Discussion

The Lanczoserrorestimation procedure is applicable to the class of first order linear ODEs with polynomial coefficients defined in the interval [0,1].The procedure is restricted to first order differential system which is not good enough. The method of Fox can handle similar

The numerical examples are presented in table 4.2 below (see Delves (1976), Davey(1980) and Conte (1996)

Table 3: Error and error estimate for example 4.3

problem of order one and of higher orders than one but not general in the scope of its application. The error estimation of Onumanyi and Ortiz gives accurate due to the idea of Tau method .The idea is not economical in terms of computing because it involves matrix inversion of at least (m+s) dimension .Our present error estimation technique is extended to the class of ODEs characterized by m+s = 4, where m and s are th e order of ODE and the number of overdetermin ationrespectively and this shows a remarkable improvement over the earlier works done by these people. on the error analysis of the Tau method as its leads to error estimation formula with wider scope of application. Also this estimate does not involve any iteration for linear problems nor matrix inversion. It is observed that perturbing the integrated error equation appears to improve the accuracy of the error estimate significantly. The results obtained in the present work demonstrate the closeness between the exact error of the tau method, thus error estimate of the τ-method is effective and reliable.

Conclusion

The integrated form of the tau method for the solution of Initial Value Problems (IVPs) involving at most four tau parameters has been presented. The error estimate is good, accurate as it closely captures the order of the approximant. This is better achieved than for the case of the differential form thus lending credence to the preference of the former. This may be due to the higher order perturbation term which the integrated formulation of tau method involves.

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