STABILITY ANALYSIS OF LOGISTIC GROWTH MODELOF ALGAE POPULATION DYNAMICS ON A WATER BODY

ABDURRAHMAN, N.
Department of Mathematics, Federal University of Technology, Minna, Nigeria
E-mail: nurat.a@futminna.edu.ng
Phone No: +234-807-790-4837

Abstract

This work analyses the stability of the equilibrium state of a logistic growth model of the Algae population dynamics on a water body thereby obtaining the critical patch length which will determine the subsistence or extinction of the water organisms.

Keywords: Logistic model, equilibrium state, phytoplankton (micro algae) diffusion, source term, environmental carrying capacity

Introduction

Algae are found in freshwater and marine environments; a few grow in terrestrial habitats. The algae are not a single, closely related taxonomic group but, instead, are a diverse assemblage of unicellular, colonial, and multicellular eukaryotic organisms. Although algae can be autotrophic or heterotrophic, most are photo autotrophs. They store carbon in a variety of forms, including starch, oils and various sugars (Microsoft Cooperation, 2003). The body of algae is called the thallus. Algae thalli range from small solitary cells to large, complex multicellular structures. Algae reproduce asexually and sexually. Some species of algae are edible and are used as gelling agent in some food. There is substantial evidence for the health benefits of algal-derived food products, but there remain considerable challenges in quantifying these benefits, as well as possible adverse effects. First, there is a limited understanding of nutritional composition across algal species, geographical regions, and seasons, all of which can substantially affect their dietary value (Mark et al, 2017). Some other types contain harmful toxins and are hazardous to health. When found in drinking water, algae can make the process of filtration more complex and costlier.

Algae are needed in aquaria and lakes to create a balanced ecosystem but can constitute problems if its growth is not controlled, (Algae Wikipedia, 2018).

A Model may be defined as a simplified or idealized descriptions or conception of a particular system, situation or process. It may be categorized according to the medium in which they are expressed, (Akinwande, 2018).

Mathematical modeling is the process of creating a mathematical representation of some phenomenon in order to gain a better understanding of that phenomenon, (Benyah, 2009). It has become an important scientific technique over the last three decades and is becoming more and more a powerful tool to solve problems arising from science, engineering, economics, industries and the society in general, (Akinwande, 2018).

The logistic population growth model assumes that environment has a carrying capacity K. This is the maximum population which the environment can sustain; the model equation is given by;

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right) \qquad p(0) = p_0 \tag{1}$$

The analytic solution of (1) is given by

$$p(t) = \frac{Kp_0}{\left(K - p_0\right)\ell^{-rt} + p_0} \tag{2}$$

This model has been successful applied to several population dynamics with a great measure of success by Akinwande (2006). The major deficiency of this model is that it gives no information regarding the age distribution of the population, hence assuming that the birth rate and death rates are independent of age. Lotka and Von Foerster model addressed this crucial question in their age dependent population model, giving rise to the use of partial differential equations as the population is now treated as depending on the two variables age *a* and time *t.*

Model Formulation

Following Lotka and von Foerster model, the exponential model suggests that the population apparently grows without bounds as *t* increases. This is unrealistic for *t*, the reasons being that for large N, one expects that the competition for living space and scarce resources tend to limit population size. To be more specific, the growth of algae be without bounds considering the losses at the patch boundary as the ocean body moves to and fro. In order to improve the exponential model, suppose the habitat can support a maximum population level *K* which is known as the environmental carrying capacity. When *N* reaches *K,* the growth rate is taken to be zero. This represents the extreme case in which the capacity for growth has been saturated. A reasonable modification of the rate *r* to account for this limited capacity is to consider a per capital growth rate that decreases as the patch density (ρ) increases such that;

$$M\left(x,t\right) = r\rho\left(x,t\right)\left[1 - \frac{\rho\left(x,t\right)}{K}\right] \tag{3}$$

Where $M(\mathrm{x},\mathrm{t})$ represents the difference in the source term and the sink term, and K is the environmental carrying capacity. If we consider the diffusion model with the intensity of diffusion V the rate of change of the plankton density with time is given as

$$\frac{\partial\rho}{\partial t} = v\frac{\partial^2\rho}{\partial t^2} + R(\mathrm{x},\mathrm{t}) \tag{4}$$

Where is the $R(\mathrm{x},\mathrm{t})$ source term which represent internal reproduction within the algae body. Putting equation (3) into model equation (4) we obtain

$$\frac{\partial\rho}{\partial t} = v\frac{\partial^2\rho}{\partial x^2} + r\rho\left(x,t\right)\left[1 - \frac{\rho\left(x,\mathrm{t}\right)}{K}\right] \tag{5}$$

Equation (5) is the model equation for the logistic population growth model

Methods of Solution

In dealing with equation (5) which is a non-linear system. Let us assume that the interchange between internal growth and loss at the patch boundary has been going on for a long time, so that a population density is eventually reached that depends only on position and not on the time at which it occurs, where ρ is the patch density, v is the intensity of diffusion, *t* is the time, Kis the environmental carrying capacity, *r* is the growth rate.

In essence the system has reached a steady state (equilibrium)with the environment. Under this condition, ρ no longer depends explicitly on time. It is a useful approximation to suppose that $\frac{\partial \rho}{\partial t}$ is zero. Partial derivative with respect to *x* becomes ordinary derivative in a single independent variable.

At equilibrium state $\rho(x,t)=\rho(x)$, (5) then simplifies to

$$v\frac{d^2\rho}{dx^2}+r\rho\left[1-\frac{\rho}{K}\right]=0 \tag{6}$$

Dividing through by *v* gives

$$\frac{d^2\rho}{dx^2}+\frac{r\rho}{v}\left[1-\frac{\rho}{K}\right]=0 \tag{7}$$

This is a second order equation

Stability Analysis

If equation (7) is written as a first order system by letting $u_1=\rho$ and $u_2=\rho'$ then

$$u_1'=u_2 \tag{8}$$

Equation (7) then becomes

$$u_2'=-\frac{ru_1}{v}\left(1-\frac{u_1}{K}\right) \tag{9}$$

The equation has equilibria at $u=\begin{pmatrix}0\\0\end{pmatrix}andu=\begin{pmatrix}K\\0\end{pmatrix}$. Considering the linearized stability analysis (Erwin, 1988) of (8) and (9). We then obtain the Jacobian determinant for the eigenvalue λ

$$\begin{vmatrix}0-\lambda & 1\\ -\frac{ru_1}{v}\left(-\frac{1}{K}\right)-\left(1-\frac{u_1}{K}\right)\frac{r}{v} & 0-\lambda\end{vmatrix}=0, \tag{10}$$

The characteristic equation is therefore given by

$$\lambda^2-\frac{ru_1}{vK}+\frac{r}{v}\left(1-\frac{u_1}{K}\right)=0 \tag{11}$$

At $u=\begin{pmatrix}0\\0\end{pmatrix}$ equation (11) becomes

$$\lambda^2+\frac{r}{v}=0 \tag{12}$$

$$\lambda^2=-\frac{r}{v} \tag{13}$$

$$\lambda=\pm i\sqrt{\frac{r}{v}} \tag{14}$$

At $u=\begin{pmatrix}K\\0\end{pmatrix}$ (11) becomes

$$\lambda^2 - \frac{r}{v} = 0 \tag{15}$$

$$\lambda^2 = \frac{r}{v} \tag{16}$$

$$\lambda = \pm\sqrt{\frac{r}{v}} \tag{17}$$

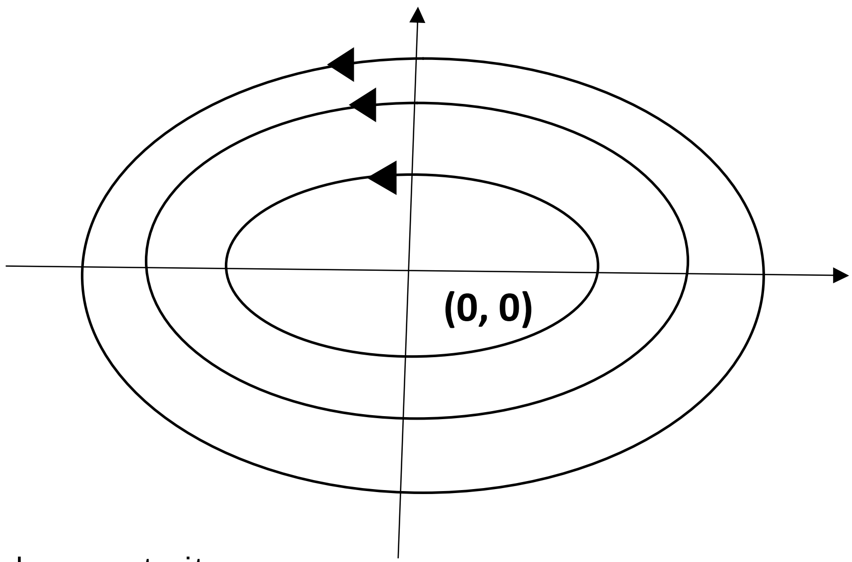

Figure 1: phase portrait

In a steady state, the boundary condition becomes $\rho(0) = \rho(L) = 0$.

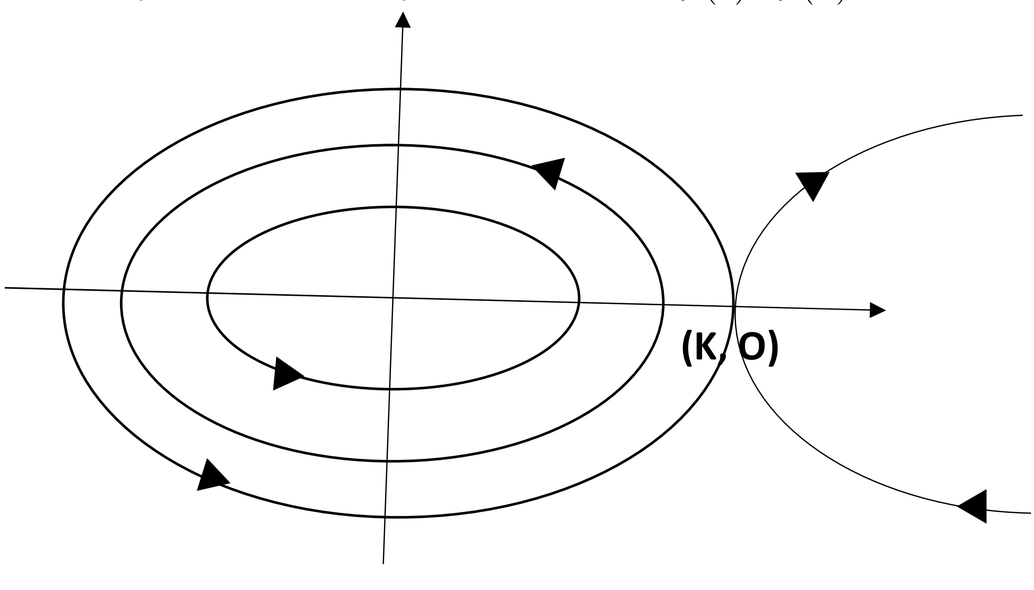

Figure 2: Phase portrait

The initial point is outside the region bounded by the separatrix (stable manifold) leading to the saddle, $u = \begin{pmatrix} K \\ 0 \end{pmatrix}$ then such a path is impossible. It follows that $\rho < K$ for all x.

Equations (14) and (17) can be represented by the phase portraits shown in figure 1 and figure 2 respectively.

In a steady state, the boundary condition becomes $\rho(0)=\rho(L)=0$. If a non-trivial solution (7) exists then it must begin and end on the vertical ρ' axis, since $\rho \geq 0$ for all *x*, this can only occur if the orbits begin on the positive ρ' axis and moves clockwise around the oval shaped path. If the initial point is outside the region bounded by the separatrix (stable manifold) leading to the saddle, $u=\begin{pmatrix} K \\ 0 \end{pmatrix}$ then such a path is impossible. It follows that $\rho < K$ for all *x.* When the orbit begins on the separatrix itself. Since the saddle is an equilibrium state it can only approach it asymptotically. (Beltrami, 1989).

This means that the orbit corresponds to an interval of infinite length and in this case the boundary condition $\rho(L)=0$ cannot be satisfied since the orbit is not able to continue onto the ρ' axis.

$$\rho'(x)=\pm\sqrt{2\left(C-U(\rho)\right)} \tag{18}$$

Let an orbit cross the horizontal axis at some $\rho_1 < K$ since the path is symmetrical about the ρ' axis. Consider only half of it as ρ goes from zero to ρ_1 it covers a distance $\frac{L}{2}$ then

$$L=2\int_0^{L/2} dx=\int_0^{\rho_1}\frac{d\rho}{\rho} \tag{19}$$

$$L=\sqrt{2}\int_0^{\rho_1}\frac{d\rho}{\sqrt{C-U(\rho)}} \tag{20}$$

We assume here that ρ' is not zero. Otherwise ρ is a constant namely zero because of the boundary conditions. If we consider C = U(ρ_1)

$$L=\sqrt{2}\int_0^{\rho_1}\frac{d\rho}{\sqrt{U(\rho_1)-U(\rho)}} \tag{21}$$

Let

$$Z=U(\rho)=\frac{r}{v}\left[\frac{\rho^2}{2}-\frac{\rho^3}{3K}\right] \tag{22}$$

$$\frac{dZ}{d\rho}=\frac{r}{v}\left[\rho-\frac{\rho^2}{K}\right] \tag{23}$$

$$L=\sqrt{2}\int\frac{d\rho}{\sqrt{U(\rho_1)-Z}}\times\frac{dZ}{d\rho} \tag{24}$$

$$L=\frac{r\sqrt{2}}{v}\int_0^{\rho_1}\left(U(\rho_1)-Z\right)^{-1/2}\times\left(\rho-\frac{\rho^2}{K}\right)d\rho \tag{25}$$

Integrating by part, we have

$$L=\frac{r\sqrt{2}}{v}\left(2\left(\rho-\frac{\rho^2}{K}\right)\times\left(U(\rho_1)-Z\right)^{1/2}-2\int_0^{\rho_1}\left(1-\frac{2\rho}{K}\right)\times\left(U(\rho_1)-Z\right)^{1/2}d\rho\right) \tag{26}$$

$$L=\frac{2r\sqrt{2}}{v}\left(\rho-\frac{\rho^2}{K}\times\left(U(\rho_1)-Z\right)^{1/2}-\frac{2}{3}\left(1-\frac{2\rho}{K}\right)\times\left(U(\rho_1)-Z\right)^{3/2}+\frac{2}{3}\int_0^{\rho_1}\left(U(\rho_1)-Z\right)^{3/2}\times\frac{2}{K}d\rho\right) \tag{27}$$

$$L=\frac{2r\sqrt{2}}{v}\left(\rho-\frac{\rho^{2}}{K}\times\left(U\left(\rho_1\right)-Z\right)^{\frac{1}{2}}-\frac{2}{3}\left(1-\frac{2\rho}{K}\right)+\frac{4}{3K}\left(U\left(\rho_1\right)-Z\right)^{\frac{5}{2}}\times\frac{2}{5}\right)_0^{\rho_1} \tag{28}$$

$$L=\left(\frac{2r\sqrt{2}}{v}\left(\rho-\frac{\rho^{2}}{K}\right)\times\left(U\left(\rho_1\right)-Z\right)^{\frac{1}{2}}-\frac{4r\sqrt{2}}{3v}\left(1-\frac{2\rho}{K}\right)\times\left(U\left(\rho_1\right)-Z\right)^{\frac{3}{2}}+\frac{16}{3K}\frac{16r\sqrt{2}}{15vK}\left(U\left(\rho_1\right)-Z\right)^{\frac{5}{2}}\right)_0^{\rho_1} \tag{29}$$

Substituting $U\left(\rho\right)$ for Z

$$L=\left(\frac{2r\sqrt{2}}{v}\left(\rho-\frac{\rho^{2}}{K}\right)\times\left(U\left(\rho_1\right)-U\left(\rho_1\right)\right)^{\frac{1}{2}}-\frac{4r\sqrt{2}}{3v}\left(1-\frac{2\rho}{K}\right)\times\left(U\left(\rho_1\right)-U\left(\rho_1\right)\right)^{\frac{3}{2}}+\frac{16}{3K}\frac{16r\sqrt{2}}{15vK}\left(U\left(\rho_1\right)-U\left(\rho_1\right)\right)^{\frac{5}{2}}\right)_0^{\rho_1} \tag{30}$$

$$L=\left(\frac{2r\sqrt{2}}{v}\left(\rho-\frac{\rho^{2}}{K}\right)\times\left(U\left(\rho_1\right)-U\left(\rho_1\right)\right)^{\frac{1}{2}}-\frac{4r\sqrt{2}}{3v}\left(1-\frac{2\rho}{K}\right)\times\left(U\left(\rho_1\right)-U\left(\rho_1\right)\right)^{\frac{3}{2}}+\frac{16}{3K}\frac{16r\sqrt{2}}{15vK}\left(U\left(\rho_1\right)-U\left(\rho_1\right)\right)^{\frac{5}{2}}\right)$$
$$-0+\frac{4r\sqrt{2}}{3v}U\left(\rho_1\right)^{\frac{3}{2}}-\frac{16r\sqrt{2}}{15vK}U\left(\rho_1\right)^{\frac{5}{2}} \tag{31}$$

$$L=\frac{4r\sqrt{2}}{3v}U\left(\rho_1\right)^{\frac{3}{2}}-\frac{16r\sqrt{2}}{15vK}U\left(\rho_1\right)^{\frac{5}{2}} \tag{32}$$

$$L=\frac{r}{v}\left(\frac{4\sqrt{2}}{3}U\left(\rho_1\right)^{\frac{3}{2}}-\frac{16\sqrt{2}}{15K}U\left(\rho_1\right)^{\frac{5}{2}}\right) \tag{33}$$

But

$$U\left(\rho_1\right)=\frac{r}{v}\left[\frac{\rho_1^{2}}{2}-\frac{\rho_1^{3}}{3K}\right] \tag{34}$$

Substituting equation (34) into equation (33) we have

$$L=\frac{r}{v}\left(\frac{4\sqrt{2}}{3}\left(\frac{r}{v}\left[\frac{\rho_1^{2}}{2}-\frac{\rho_1^{3}}{3K}\right]\right)^{\frac{3}{2}}-\frac{16\sqrt{2}}{15}\left(\frac{r}{v}\left[\frac{\rho_1^{2}}{2}-\frac{\rho_1^{3}}{3K}\right]\right)^{\frac{5}{2}}\right) \tag{35}$$

Equation (35) gives the critical value of the patch which has to be maintained for the phytoplankton to still be in existence. It will be observed that the patch length varies directly as the growth rate (*r)* and inversely as the intensity of diffusion (*v*). This means that length of the patch increases with increased growth rate and decreased intensity of diffusion.

Conclusion

Equation (35) gives the critical value of the length of phytoplankton that should be maintained by fish farmers to avoid going into extinction as they supply some essential nutrients needed in the fishpond.

References

Akinwande, N. I. (2018). Introductory notes on biomathematics a paper presented at the third Workshop on mathematical modelling, Department of mathematics, University of Nigeria, Nsukka.

Akinwande, N. I., (2006). *On the application of differential equations in the mathematical modelling of population dynamics.* Book of Reading, 2nd Annual conference of the school of science education FUT Minna. PP 77-84.

Akinwande, N. I., (2004). *Introduction to fourier series and transform.* Ibadan: Associated Book Maker Nigeria Ltd.

Beltrami, E. (1989). *Mathematical for dynamics modelling.* London: Academic Press Inc.

Benyah, F. (2005). *Introduction to Mathematical modelling.* 7th Regional college on modelling, Simulation and Optimization Cape Coast Ghana.

Erwin, K. (1998). *Advanced Engineering Mathematics.* London: Wiley Eastern limited

Google Dictionary https://en.wikipedia.org/wiki/Algae

Mark, L., Wells, P. P., James, S., Craigie, J. A., Raven, M., Katherine, A. G., Smith, M. E. C., & Susan, H. B. (2017), Algae as nutritional and functional food sources: Revisiting our understanding. *Journal of Applied Phycology,* 29(2), 949–982. doi: [10.1007/s10811-016-0974-5]